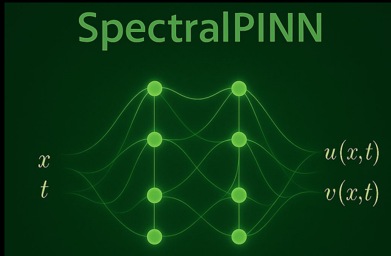
Teukolsky by Design: A Hybrid Spectral-PINN solver for Kerr Quasinormal Modes

Alexandre M. Pombo



Introduction: Quasinormal modes

- Gravitational wave detections are a window to explore compact objects
- These are ideal probes of fundamental and dark-sector physics
- Quasi-normal modes encode stability and the ringdown information
- The resulting GW carries information about the merging objects and the final remnant

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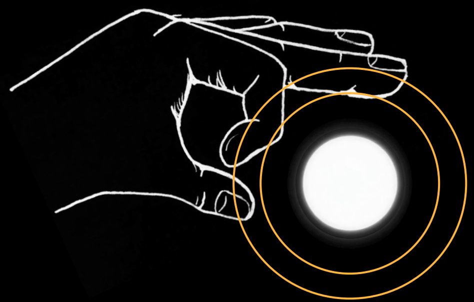

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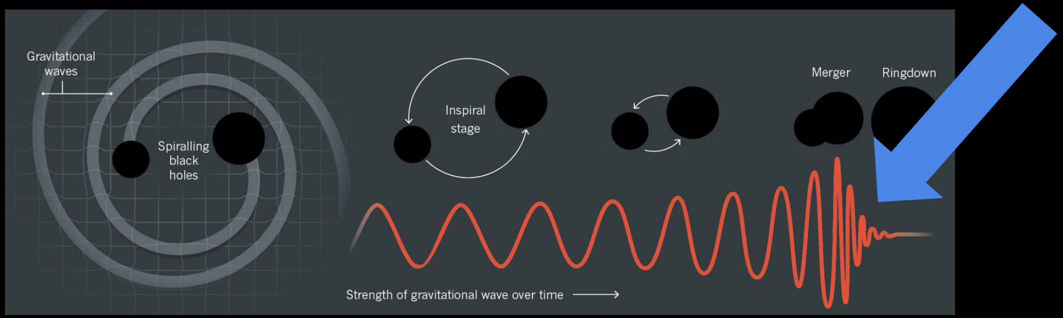

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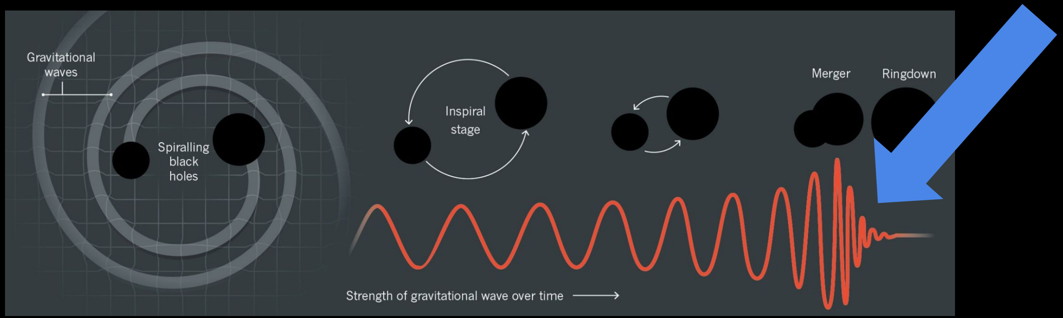

- QNM computation is challenging, leaving many models unexplored.

Kerr BH: Teukolsky equation

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}$$
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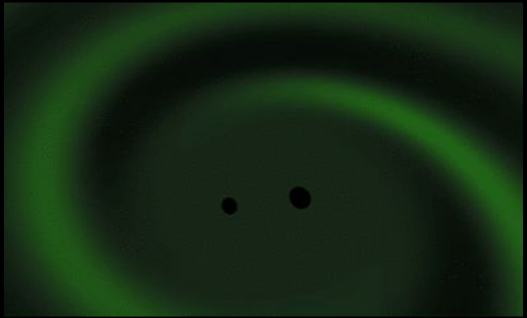

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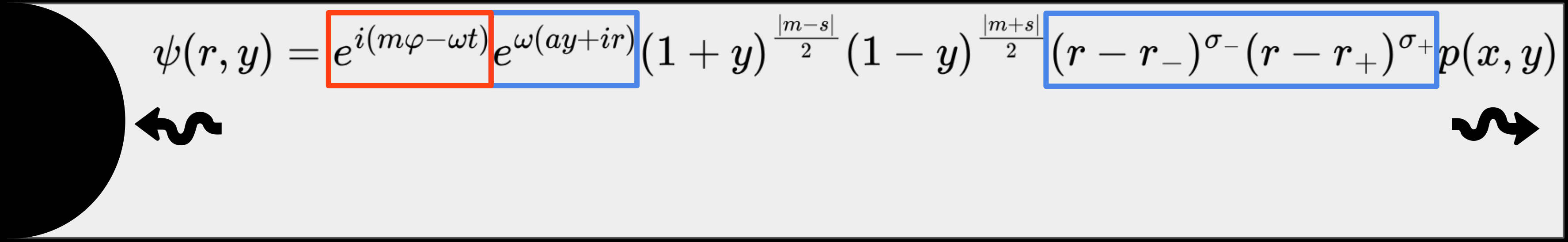

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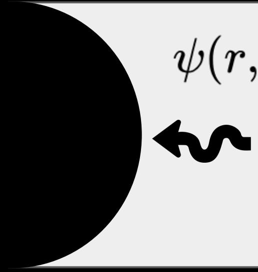

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$$p(x,y) = f(x)g(y)$$

Normalization conditions

- The Teukolsky QNM are a homogeneous linear eigenvalue problem
- QNM amplitudes are fixed by the initial data rather
- Once (ω, Λ) is fixed the equations admit: $(f, g) \rightarrow (C\,f, C^{-1}\,g)$ □ $p \rightarrow C\,p$
- Following Leaver's original construction

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Hard Normalization

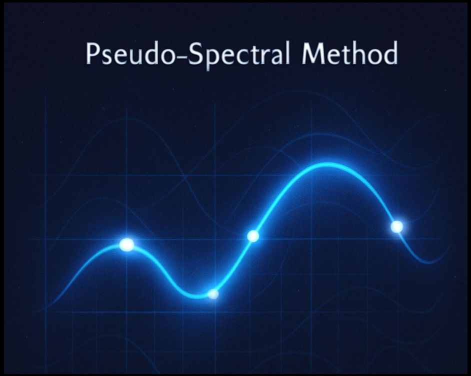
$$f(x) = \left(e^{x-1} - 1\right)\mathcal{F}(x) + 1$$

$$g(y) = \left(e^{y+1} - 1\right)\mathcal{G}(y) + 1$$

$$p(x,y) = 1 + \left[\mathcal{P}(x,y) - \mathcal{P}(1,-1)\right]$$

Pseudo-Spectral Method

$$p(x,y) = \sum A_{i,j}\, T_i(x)\, T_j(y)$$



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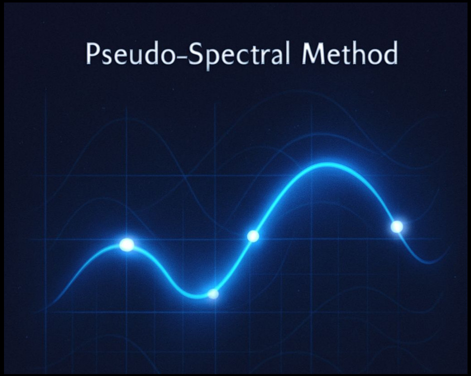
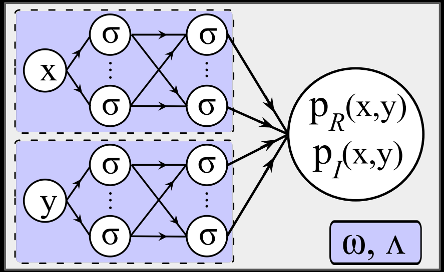
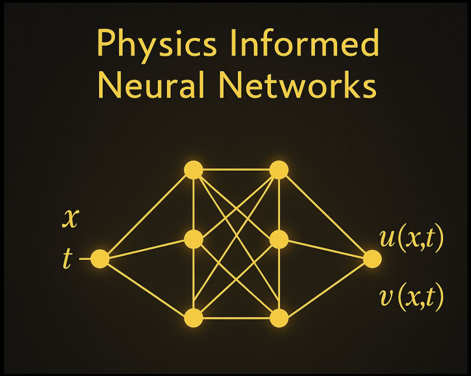


Physics Informed Neural Networks





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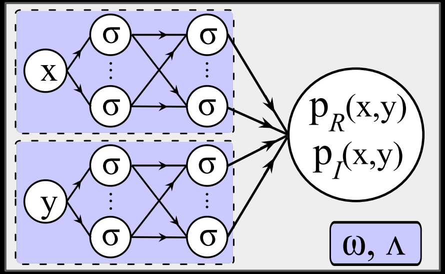

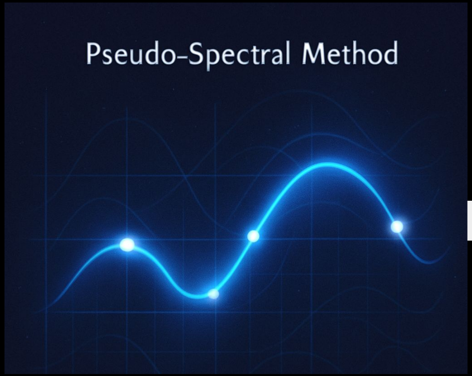

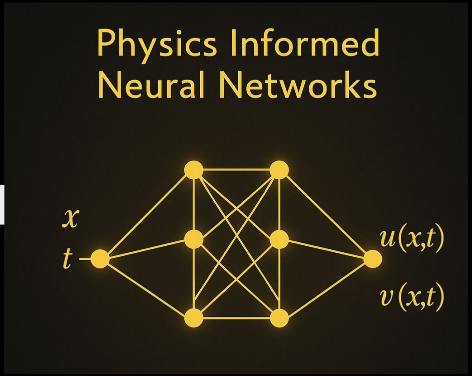

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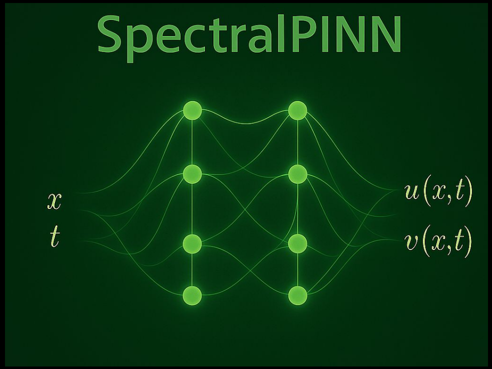

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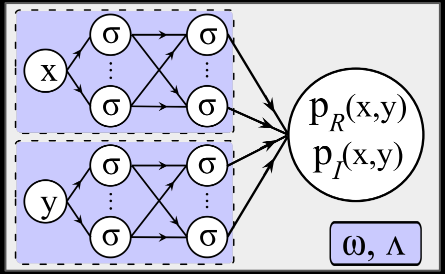

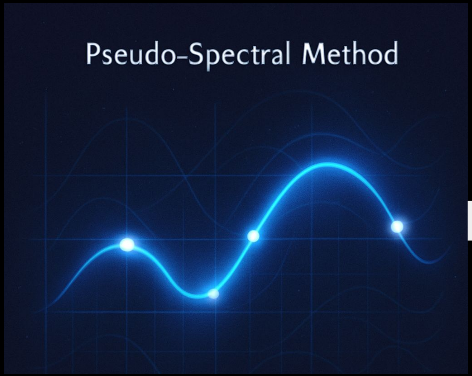

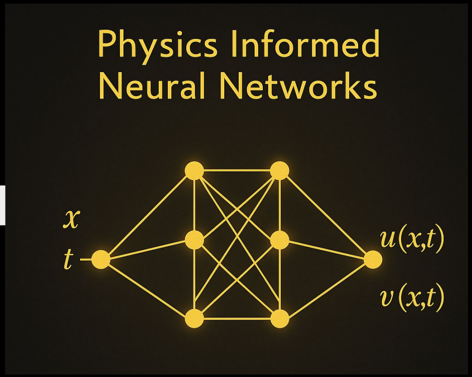

SpectralPINN

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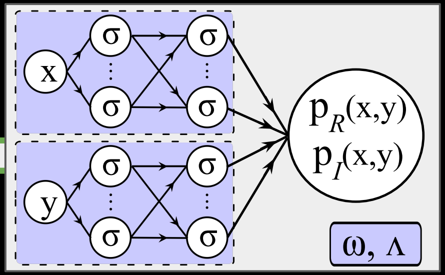
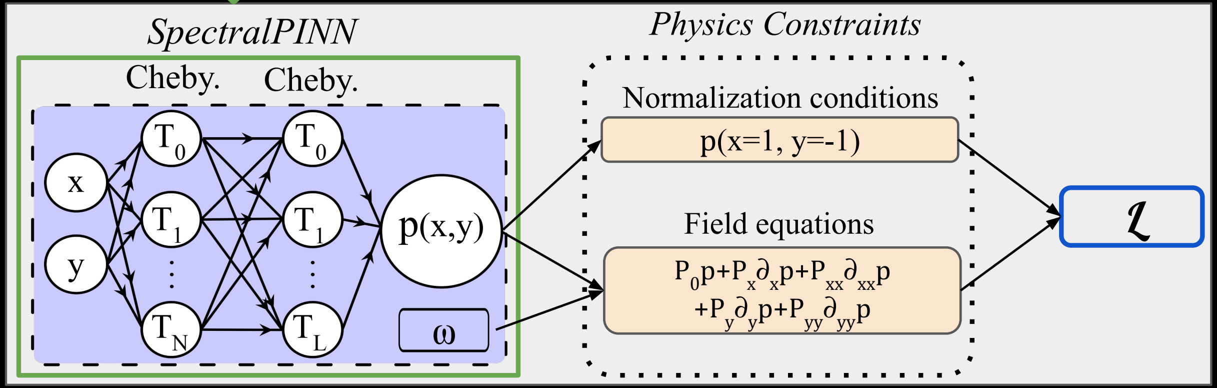
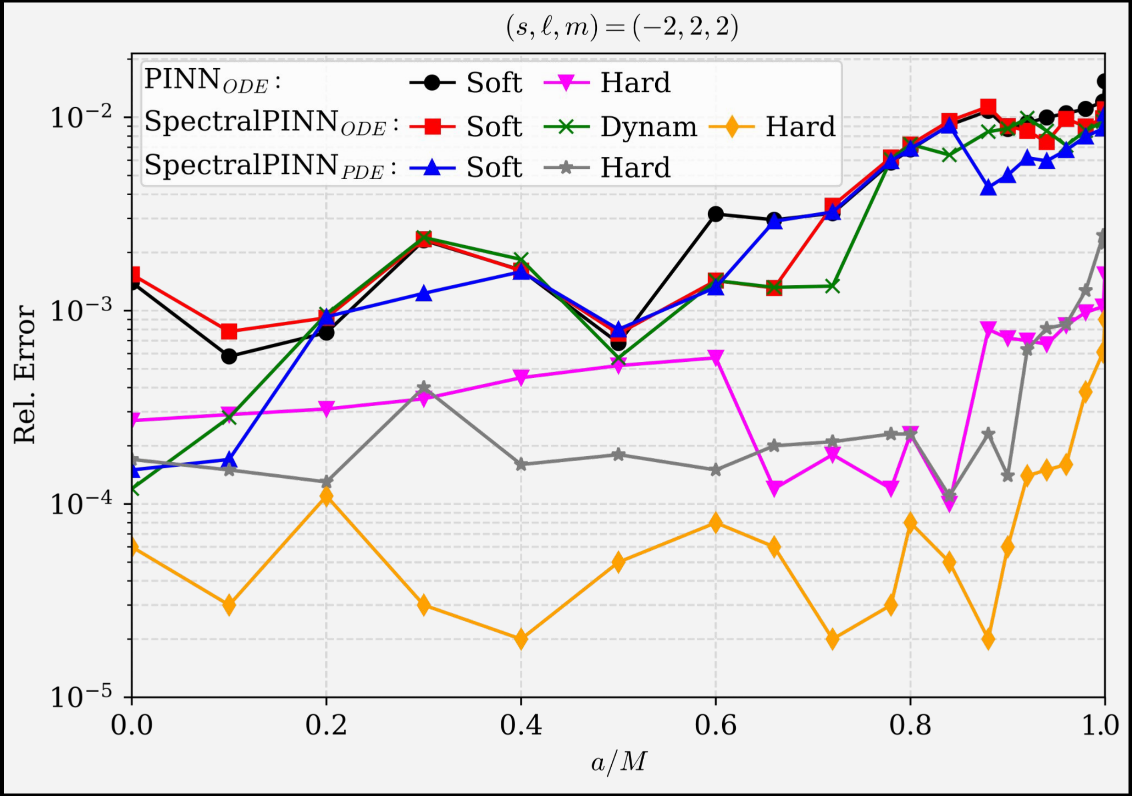




Results

Results: Numerical accuracy



Results: Quadrupolar deviation

- The SpectralPINN formulation is able to solve the 2D PDE
- Perturbations of Kerr that spoil the separability of Teukolsky equation
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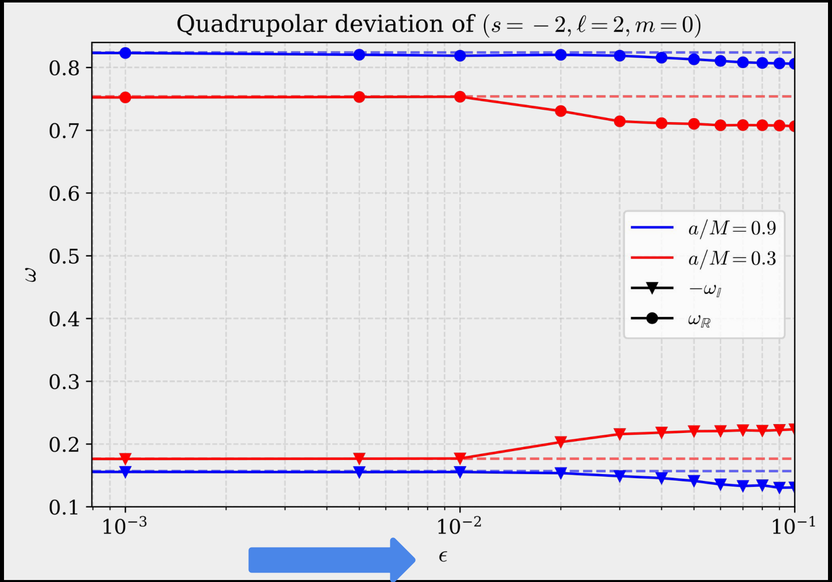

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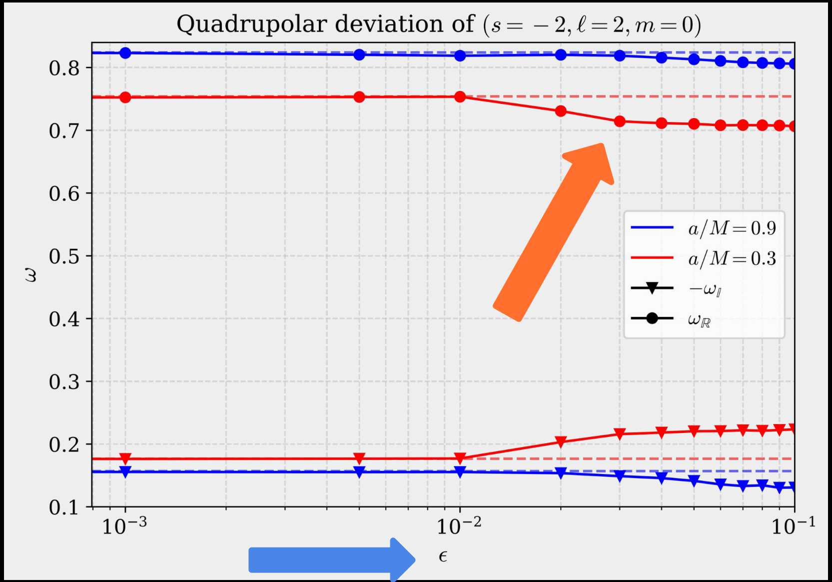
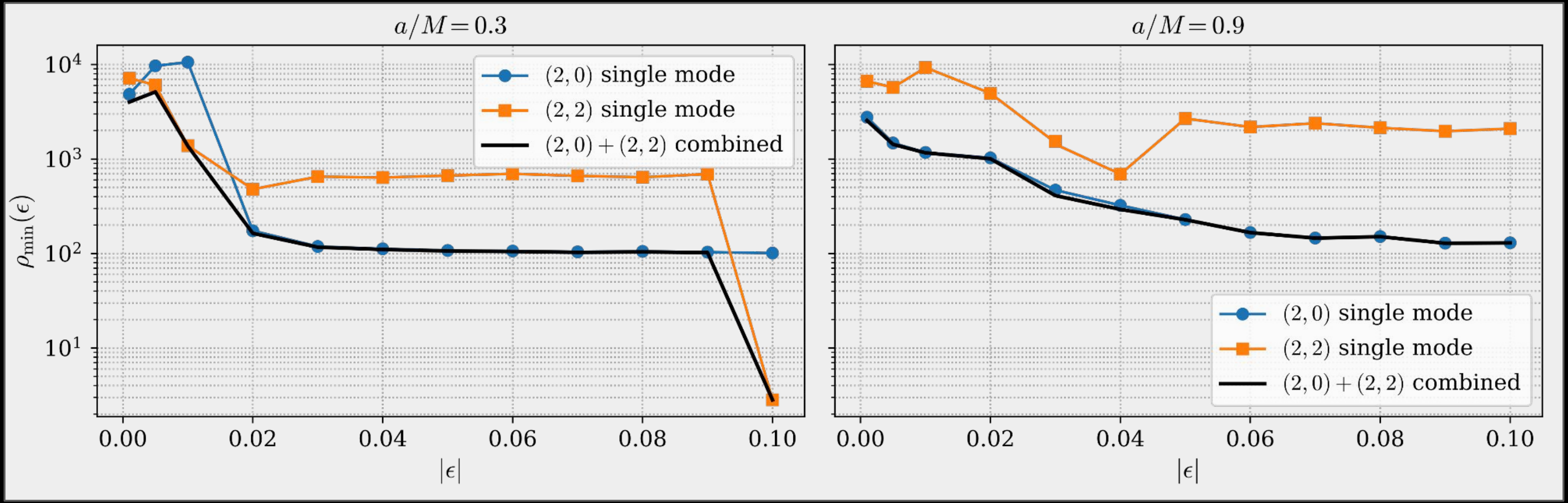


Results: Einstein Telescope



Conclusion

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Quasi-Normal Modes

SpectralPINN

Recent and near future observations

Obrigado !

Thanks !

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Thank you!

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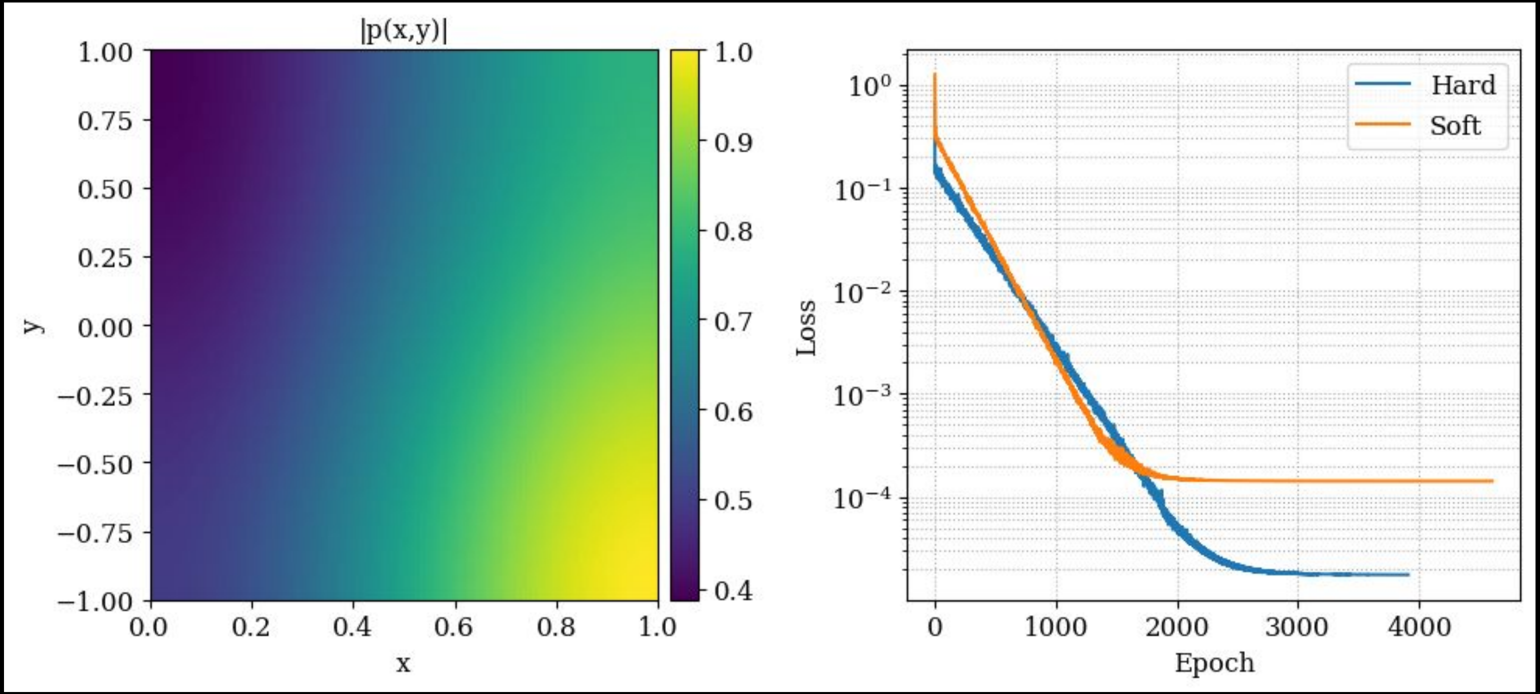
Extra quirks: Training



Extra quirks: Basis number

	$N \times L$						
Eq. type	10×10	15×10	10×15	15×15	20×20	30×30	50×50
ODE	0.714	0.109	0.156	0.012	0.006	0.002	0.429
PDE	0.920	0.234	0.280	0.071	0.052	0.011	0.600

Extra quirks: Training improvements

Scheme	Eq. type	Base	+$\mathbb{C}$-dtype	+HAdamD	+Alt. Sched.
PINN	ODE	0.138	0.113	0.069	0.050
SpectralPINN	ODE	0.131	0.126	0.016	0.002
SpectralPINN	PDE	0.142	0.134	0.058	0.011