# Zeldovich vacuum energy density and strong gravity with running gravitational constant

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**Abstract**

A static geometry is studied in this paper. The source of curvature is given by an anisotropic stress tensor. The null and timelike radial geodesics are investigated and found to represent hyperbolae, but with different accelerations. Due to the very high acceleration ($a \approx 10^{34} cm/s^2$) the massless particle reach very quickly the velocity $c$. Some numerical examples are given, emphasizing the strong curvatures near $r = 1/a$. The stress tensor creating the curvatures has a positive trace and represents an imperfect fluid.

## 1 Introduction

The scale dependence of the gravitational action implies that there should be very large fluctuations of the metric and of the spacetime on length scales of order the Planck length, or less [1], with high curvatures. One is thus led to a picture of "spacetime foam" in which the dominant contribution to the path integral comes from metrics with about one unit of topology per Planck volume. However, everyday observations indicate that spacetime is nearly flat when viewed on standard scales [1].

We expect the foamlike structure to dominate all of spacetime. However, for the purposes of interpretation it is convenient, to think that there is an asymptotic region in which the fluctuations and interactions are turned off and where the fields can be regarded as propagating freely in a background metric which is quasi-flat [2]. This allows one to interpret the asymptotic in and out fields in terms of particles and to define S-matrices which give the transformations from the initial to the final situations. In quantum gravity it is difficult to define gauge-invariant observable quantities, excepting in asymptotic regions.

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Motivated by the Ref.[2], we are looking in this paper for a different recipe to get an almost flat background line-element for a foamlike-dominated geometry. Our purpose is to introduce a metric where the geometry is nearly flat when $r >> 1/a$ ($a$ is a constant acceleration), having high curvatures very close to $1/a$.

The geometric units $c = G = \hbar = 1$ are used, unless otherwise specified, where $c$ is the velocity of light from Special Relativity, $G$ is the Newton constant and $\hbar$ is Planck's constant.

# 2 Static spacetime

Let us consider the following static metric

$$ds^2 = -dt^2 + \frac{a^2r^2}{a^2r^2 - 1}dr^2 + r^2 d\Omega^2, \tag{2.1}$$

with $r > 1/a$, $a$ is a positive constant, having units of acceleration, $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$ stands for the unit 2-sphere. The above restriction is necessary for to preserve the signature $(-,+,+,+)$, otherwise $r$ becomes a timelike coordinate. The geometry (2.1) is curved and tends to become Minkowskian for $ar >> 1$. It can be shown that the Ricci and Kretschmann scalars are given by

$$R^b_{\ b} = -\frac{2}{a^2r^4}, \quad K = \frac{12}{a^4r^8} \tag{2.2}$$

One notices from (2.1) that an observer set at $r = const.$ and having the 4-velocity $u^a = (1,0,0,0)$ is geodesic because $g_{tt} = -1$. However, if we consider $U^a = (ar, (a^2r^2 - 1)/ar, 0, 0)$, with $U^aU_a = -1$, the covariant acceleration $a^b = U^a\nabla_a U^b$ looks like

$$a^b = \left(\frac{a^2r^2 - 1}{r}, \frac{a^2r^2 - 1}{r}, 0, 0\right), \tag{2.3}$$

and its modulus is given by

$$A = \sqrt{g_{ab}a^a a^b} = \frac{\sqrt{a^2r^2 - 1}}{r} \tag{2.4}$$

whici tends to $a$ when $ar >> 1$. In addition, $A$ is vanishing when $r \to 1/a$.

# 3 Anisotropic stress tensor

One looks now for the energy-momentum tensor to be inserted on the r.h.s. of Einstein's equations $G_{ab} = 8\pi T_{ab}$ and which represents the source of curvature for the spacetime (2.1). The mixed components of the Einstein tensor are given by

$$T^t_{\ t} = -T^r_{\ r} = T^\theta_{\ \theta} = T^\phi_{\ \phi} = \frac{1}{8\pi a^2 r^4}, \tag{3.1}$$

with all the other components being zero. One observes that the above energy-momentum tensor corresponds to an anisotropic fluid because radial and transversal pressures are different. Therefore, $T^a_{\ b}$ can be written as

$$T^a_{\ b} = (\rho + p_t)u^a u_b + p_t g^a_{\ b} + (p_r - p_t)n^a n_b, \tag{3.2}$$

where $\rho$ is the energy density of the fluid, $p_r$ is the radial pressure, $p_t$ are the transversal pressures, $n^a$ is a unit spacelike vector orthogonal to $u^a$ with

$$n^b = \left(0, \frac{\sqrt{a^2r^2 - 1}}{ar}, 0, 0\right). \tag{3.3}$$

We have $u_a n^a = 0,\ u_a u^a = -1,\ n_a n^a = 1$. From (3.2) it is clear that

$$\rho = T^a_{\ b} u_a u^b = -T^t_{\ t} = -\frac{1}{8\pi a^2 r^4},\quad p_r = T^r_{\ r} = -\frac{1}{8\pi a^2 r^4},\quad p_t = T^\theta_{\ \theta} = \frac{1}{8\pi a^2 r^4} \tag{3.4}$$

It is worth noting that, albeit $\rho$ is negative, the trace of the stress tensor is positive

$$T^a_{\ a} = -2\rho = \frac{1}{4\pi a^2 r^4}. \tag{3.5}$$

Keeping in mind (3.4), (3.2) can be written as

$$T^a_{\ b} = -\rho\delta^a_{\ b} + (\rho + p_r)n^a n_b. \tag{3.6}$$

# 4 Radial geodesics

Our next purpose is to find the geodesics on the r-direction for the geometry (2.1), using the Lagrangean method. We have

$$L = -\left(\frac{ds}{d\lambda}\right)^2 = \frac{1}{2}\left(\dot{t}^2 - \frac{a^2r^2}{a^2r^2 - 1}\dot{r}^2\right) \tag{4.1}$$

where $\lambda$ is the affine parameter along the geodesic, with $\theta, \phi$-const.. A dot means derivative w.r.t $\lambda$. The Euler-Lagrange equations are written as

$$\frac{\partial L}{\partial t} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{t}}\right) = 0, \quad \frac{\partial L}{\partial r} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0, \tag{4.2}$$

which leads to

$$\dot{t} = E, \quad \dot{r}^2 - r(a^2r^2 - 1)\ddot{r} = 0, \tag{4.3}$$

where $E$ is the energy per unit mass of the massive test particle.

i) **Null radial geodesics**

For null geodesics we have, from (4.1) that $L = 0$, that is

$$\dot{t}^2 - \frac{a^2r^2}{a^2r^2 - 1}\dot{r}^2 = 0, \tag{4.4}$$

which is easily integrate and it gives us a hyperbolic trajectory for the test null particle

$$r(t) = \frac{1}{a}\sqrt{1+a^2t^2}, \qquad r(0) = 1/a. \tag{4.5}$$

It is not difficult to show that the second equation (4.3) is verified identically. From (4.5) the velocity appears as

$$v(t) = \frac{at}{\sqrt{1+a^2t^2}} \leq 1, \tag{4.6}$$

a well known expression for the uniformly accelerated particle.

ii)**Timelike radial geodesics**

Apart from the two eqs.(4.3), we have in addition that

$$\dot{t}^2 - \frac{a^2r^2}{a^2r^2-1}\dot{r}^2 = 1, \tag{4.7}$$

From (4.7) and (4.3) one obtains

$$\frac{ar}{\sqrt{a^2r^2-1}}\dot{r} = \sqrt{E^2-1}, \qquad \dot{r} = dr/d\tau, \tag{4.8}$$

where $\tau$ is the proper time. Using now the first equation (4.3) we find that

$$r(t) = \frac{1}{a}\sqrt{1+\frac{a^2(E^2-1)t^2}{E^2}}, \tag{4.9}$$

where $a\sqrt{E^2-1}/E$ plays the role of an effective acceleration. $E^2-1$ is in fact $E^2-m^2$, where $m$ is the particle rest mass. When $m=0$, we get exactly the geodesics for the massless particle (4.5). We have again that the second equation (4.3) is verified for the massive particle too.

It is interesting that both type of geodesics are represented by a hyperbola, with a smaller acceleration for the timelike ones. Moreover, they were deduced by means of simple mathematical calculations.

# 5 Applications in microphysics

Our goal is to apply the above model in microphysics. We see that the null geodesic $x(t) = (1/a)\sqrt{1+a^2t^2}$ becomes a straight line $x=t$ when $a$ is very large or $at >> 1$. For example, if we take a proton, its maximum acceleration [3, 4, 5] should be $a_{max} = c^2/\lambda_c \approx 10^{34} cm/s^2$, where $\lambda_c = \hbar/cm_p$ is the Compton wavelength and $m_p$ is the proton mass. From now on we assume that the acceleration $a$ used by now in the expression of the null geodesic is equal to $a_{max}$. In other words, in our view for very short time intervals, taken from the moment of emission, a massless particle moves hyperbolically, with a very large acceleration. Therefore, a massless particle cannot get the velocity $c$ instantaneously but it undergoes an accelerated motion (with very large acceleration) for a very short time interval and then the velocity becomes $c$. Same is valid for a massive object which cannot pass from rest to uniform motion without a previous acceleration. The origin of acceleration is supposed to be given by the quantum vacuum fluctuations (the so called spacetime foam [1]). The null geodesics along the r-direction become $r = t$, when $ar >> 1$ and the metric (2.1) becomes Minkowskian.

Dealing with the microscopic world (namely, with the physical systems with their mass $m << m_P$, where $m_P$ is the Planck mass), we shall use the strong gravitational constant [6] $G_s = c\hbar/m^2$. If one use the proton mass instead of $m$, we obtain $G_s = c\hbar/m_p^2 \approx 3 \cdot 10^{31}$, in CGS units. That is justified because, as we noticed already, the geometry (2.1) deviates from the flat geometry when $a^2r^2$ is close to unity from above. Note that Zeldovich [7] and the authors of [8] proposed a connection between Cosmology (radius of the Universe) and the proton radius (see also [9]).

The Zeldovich vacuum energy density is given by

$$\epsilon_{vac} = \frac{Gm_p^2}{\lambda_c V} \approx \frac{Gm_p^6 c^4}{\hbar^4} \propto m_p^6, \tag{5.1}$$

where $V$ is the proton volume and $\epsilon_{vac} \approx 10^{-2} erg/cm^3$. In the strong gravity regime, with $G_s = c\hbar/m_p^2$, one obtains

$$\epsilon_{vac} = \frac{G_s m_p^2}{\lambda_c V} \approx \frac{m_p^4 c^5}{\hbar^3} \propto m_p^4, \tag{5.2}$$

and we get $\epsilon_{vac} \approx 10^{37} erg/cm^3$, a more realistic value of the quantum vacuum energy density. A similar dependence of $\epsilon_{vac}$ on the particle mass has recently been obtained by Andre LeClair [10], who states that $\epsilon_{vac} \propto m_z^4$, where $m_z$ is the physical mass of the zeron, as he coined the vacuum particle. Moreover, in his model $G$ is a function of the energy scale, something similar to our situation.

If one computes now the energy resulting from the energy density $\rho$ in our strong gravity regime, we have

$$W = \int_{1/a}^{\infty} \rho\sqrt{-g} d^3r = -\frac{1}{2a} \int_{1/a}^{\infty} \frac{dr}{r\sqrt{a^2r^2 - 1}} \tag{5.3}$$

With a change of the variable, $y = \sqrt{a^2r^2 - 1}$, Eq.(5.3) gives us $W = -\pi/4a$. When we introduce all fundamental constants, one obtains

$$W = -\frac{\pi c^2}{4a} \cdot \frac{c^4}{G_s} = -\frac{\pi c^6}{4a} \cdot \frac{m_p^2}{\hbar c} = -\frac{\pi}{4} m_p c^2, \tag{5.4}$$

with $|W|$ of the order of proton rest energy. Of course, the fact that $W < 0$ comes from the negative sign of $\rho$.

If one looks now for the value of the Komar energy [11] $W_K$, one finds that $W_K = 0$ because its integrand is proportional to $\rho + p_r + 2p_t$, which is vanishing in our case.

Let us take an example related to the velocity (4.6). With $a = 10^{34} cm/s^2$ and taking $t = 10^{-23} s$, $v = 3 \cdot 10^{11}/\sqrt{109} \approx 2.87 \cdot 10^{10} < c$, but however, close to $c$, even though the time interval was very short. That is a consequence of the huge value of acceleration.

Another example refers to the radial pressure from (3.4). When $p_r$ is written in full, one obtains

$$p_r = -\frac{c^8}{8\pi G_s a^2 r^4}, \tag{5.5}$$

where $G_s$ is the above mentioned Newton's constant in microphysics (strong gravity). If we compute $p_r$ outside a proton and very close to it (at $r \approx 10^{-13} cm$) and with $a = 10^{34} cm/s^2$, we get $p_r \approx -10^{35} erg/cm^3$, which is (minus) the bag constant B [6], well known from QCD. In other words, the interior and exterior pressures match at the proton surface. Instead, if one considers $r = 10^{-8} cm$, we get $p_r \approx -10^{15} erg/cm^3 = -10^{-9} erg/A^3$.

# 6 Conclusions

The properties of a static spacetime with spherical symmetry, used in microphysics, are investigating in this paper. The stress tensor (source of gravity) corresponds to an anisotropic fluid, has a positive trace and negative energy density and radial pressure but positive transversal pressures. The null and timelike radial geodesics are computed. For both of them the trajectories are hyperbolae, but with different accelerations. Due to the very high acceleration ($a \approx 10^{34} cm/s^2$), the massless particles reach very quickly the velocity $c$.

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