Quasinormal modes of Kerr black holes

Leart Sabani
In collaboration with Rico Lo and Vítor Cardoso

Center of Gravity,
Niels Bohr Institute

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KØBENHAVNS UNIVERSITET
strong. niels bohr institute
THE CENTER OF GRAVITY
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Quick Motivation

→ The ringdown phase of black hole mergers is well described by quasinormal modes.

→ It is important to know the precise predicted frequencies of these modes for tests of General Relativity.

→ New method, to compare with existing results from fx. Leaver's method.

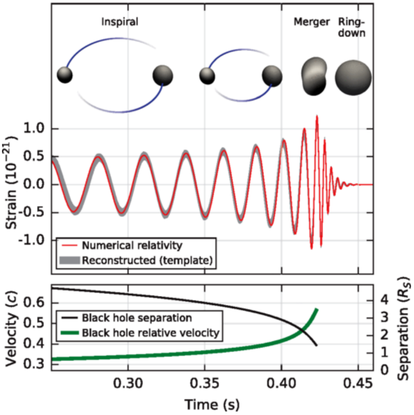

Figure 1: Credit: Phys. Rev. Lett. 116, 061102 (2016).

Perturbations in the Kerr Background

Linear gravitational perturbations of the Kerr metric are described by the **Teukolsky equation**. The radial equation can be written in the form

$$\frac{d^2}{dr_*}Y + \left[\omega^2 + V_Y(r)\right]Y = 0$$

Where V_Y is a long ranged potential: $V_Y \sim 1/r$.
Difficult for numerical integration of the equation.

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The Generalized Sasaki-Nakamura transformation (GSN) converts the long-range potential behavior of the Teukolsky equation into a short-ranged one.

$$\left[\frac{d^2}{dr_*^2} - {}_s\mathcal{F}_{\ell m\omega}(r)\frac{d}{dr_*} - {}_s\mathcal{U}_{\ell m\omega}(r)\right] {}_sX_{\ell m\omega}(r) = 0$$

Here $\mathcal{U} \to 1/r^2$

Direct Integration of the GSN Equation

Quasinormal modes are defined by imposing **outgoing** boundary conditions at infinity and **ingoing** boundary conditions at the outer horizon:

$$X \to \begin{cases} e^{-i(\omega - m\Omega_H)r_*}, & r_* \to -\infty \\ e^{+i\omega r_*}, & r_* \to +\infty \end{cases}$$

Since QNM frequencies ω are **complex**, these solutions grow or decay exponentially in the asymptotic regions, making direct numerical integration unstable.

Complex Scaling Transform

To control the exponential behavior, we deform the integration contour into the complex r_* plane:

$$r_* = r_*^{\mathrm{mp}} + \rho\, e^{i\beta}, \quad \rho \in \mathbb{R}.$$

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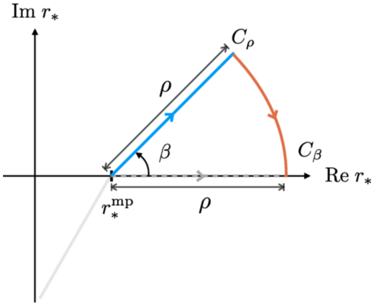

Figure 2: Complex deformation of the r_* contour.

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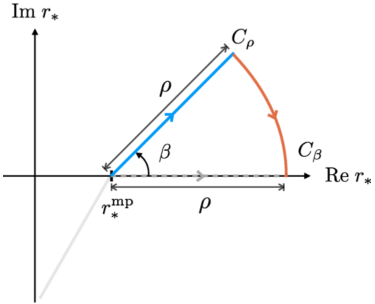
Under this transformation, the asymptotic behavior becomes

$$e^{i\omega r_*} \;\rightarrow\; e^{i\omega\rho e^{i\beta}},$$

which is **purely oscillatory** if

$$\mathrm{Im}\big(e^{i\beta}\omega\big) = 0 \quad \Rightarrow \quad \beta = -\arg\omega.$$

This choice removes exponential growth and allows for stable numerical integration.



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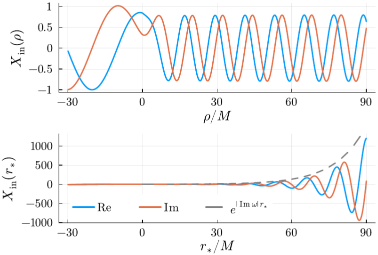
Complex Scaling Transform

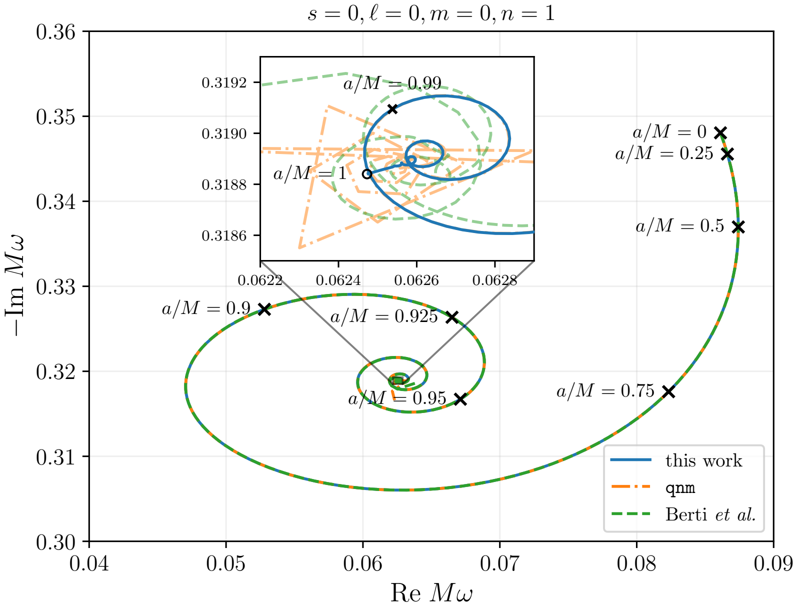
After the contour deformation $r_* = \rho e^{i\beta}$, the GSN equation becomes

$$\left[\frac{d^2}{d\rho^2} - e^{i\beta}\mathcal{F}\frac{d}{d\rho} - e^{2i\beta}\mathcal{U}\right] X = 0.$$

- → Complex scaling rotates exponential growth into oscillatory behavior.
- → The equation is integrated from **both boundaries** toward a matching point.
- → QNMs are found by *shooting* for frequencies where the two solutions match.



Calculated Quasinormal Modes



Conclusion

- → We have developed a new method to calculate quasinormal modes of Kerr black holes based on the Generalized Sasaki–Nakamura formalism.
  - ⋆ Very stable, and works for large black hole spin and angular number ℓ.
  - ⋆ Fast, with very high numerical precision.
- → The code is publicly available on GitHub.
  - ⋆ Works for scalar, vector and gravitational perturbations.

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