Lagrangian Reverse Engineering for Regular Black Holes

A. Bokulić*, E. Franzin, T. Jurić, I. Smolić

Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, Portugal

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Regular BHs

Bardeen solution [J. M. Bardeen; Proceeding of GR5 (1968)]

- metric: $ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2$,

$$f(r) = 1 - \frac{2\mu r^2}{(r^2 + g^2)^{3/2}}$$

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- corresponding nonlinear electrodynamics Lagrangian (NLE):
[E. Ayón-Beato, A. García; Phys.Lett. B493 (2000)]
$$\mathcal{L} = -\frac{3\mu}{g^3}\left(\frac{g\sqrt{2\mathcal{F}}}{2+g\sqrt{2\mathcal{F}}}\right)^{5/2}$$
- g - magnetic charge, μ - mass

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Regularisation within classical theory?

Overview of NLE

- NLE Lagrangian density: $\mathcal{L}(\mathcal{F}, \mathcal{G})$

  Electromagnetic invariants: $\mathcal{F} = F_{ab}F^{ab}$, $\mathcal{G} = F_{ab} \star F^{ab}$

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- Maxwell's weak field (MWF) limit:

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- QED weak field (QEDWF) limit:

  $\mathcal{L} = -\mathcal{F}/4 + \kappa(4\mathcal{F}^2 + 7\mathcal{G}^2) + o(\mathcal{H}^2)$, as $\mathcal{H} \to 0$,
  $\kappa = \alpha^2/(360 m_e^4)$

  motivated by Euler-Heisenberg effective theory

  [W. Heisenberg, H. Euler; Z. Phys. 98 (1936)]

Euler-Heisenberg Lagrangian

1-loop QED correction

$$\mathfrak{L} = \frac{1}{2}\left(\mathfrak{E}^2 - \mathfrak{B}^2\right) + \frac{e^2}{h c}\int_0^\infty e^{-\eta}\,\frac{\mathrm{d}\eta}{\eta^3}\left\{ i\eta^2 (\mathfrak{E}\mathfrak{B}) \cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) + \mathrm{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) - \mathrm{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}\left(\mathfrak{B}^2 - \mathfrak{E}^2\right)\right\}$$

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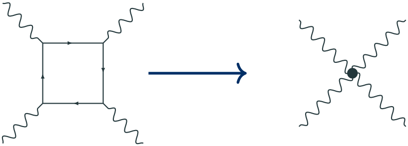
Effective Lagrangian:



$\omega \ll m_e$

$$\mathcal{L}^{(\mathrm{EH})} = -\frac{1}{4}\mathcal{F} + \frac{\alpha^2}{360 m_e^4}\left(4\mathcal{F}^2 + 7\mathcal{G}^2\right) + \mathcal{O}(\alpha^3)$$

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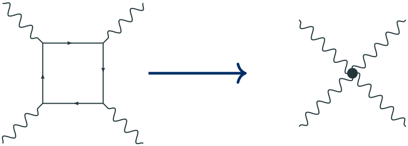

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$\gamma\gamma \to \gamma\gamma$ scattering, birefringence...

Constraints on BH regularisation

No-go results

- BHs with $Q \neq 0$ sourced by $\mathcal{L}(\mathcal{F}, \mathcal{G})$ obeying the MWF limit are not regular

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Motivation: can we find a better model?

[A. Bokulić, E. Franzin, T. Jurić, I. Smolić; Phys. Lett. B 854 (2024)]

Reverse-engineering procedure

Idea: from a chosen **regular** metric, reconstruct $\mathcal{L}(\mathcal{F}, \mathcal{G})$

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Static, spherically symmetric BHs:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m(r)}{r}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{2m(r)}{r}}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

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Generalised Maxwell's equations:

$$B_r = \frac{P}{r^2}, \quad E_r(\partial_{\mathcal{F}}\mathcal{L}) - B_r(\partial_{\mathcal{G}}\mathcal{L}) = -\frac{Q}{4r^2}$$

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Einstein's field equations:

$$-\frac{m'(r)}{r^2} = \mathcal{L} - \frac{QE_r}{r^2}, \quad -\frac{m''(r)}{2r} = \mathcal{L} - \frac{4P^2}{r^4}(\partial_{\mathcal{F}}\mathcal{L}) - \mathcal{G}(\partial_{\mathcal{G}}\mathcal{L})$$

Reverse-engineering procedure

Simplification:

- $E_r = 0$ and $Q = 0$
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Key equation:

$$\mathcal{J}(\mathcal{F}) = -\frac{m'(r)}{r^2}\bigg|_{r=(2P^2/\mathcal{F})^{1/4}}$$

Solution

Conditions on $m(r)$:

1. the existence of a BH horizon
2. $\mathcal{L}$ has to satisfy the QEDWF limit
3. the invariants R, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$ are bounded

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Reconstructed theory:

- $\mathcal{J}(\mathcal{F}) = -\frac{1}{4}\mathcal{F} + \frac{5s}{8P^4}\mathcal{F}^2 + \mathcal{O}(\mathcal{F}^{11/4})$ as $\mathcal{F} \rightarrow 0^+$
- mass-charge relation: $M \sim P^{3/2}$

Summary and open questions

Regular BH solution:

- $\mathcal{L}$ has correct weak field limit ☺
- magnetically charged 😐
- M and P are related ☹

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Questions:

- stability, formation mechanism...
- regular dyonic BHs
  [A. Bokulić, T. Jurić, I. Smolić; arXiv:2510.23711 [gr-qc] (2025)]
- fine tuning of mass and charge: Ivica's talk

Thank you for your attention!

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