Black holes in the external Bertotti-Robinson-Bonnor-Melvin electromagnetic field

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Introduction & Motivations

- Black holes observed in the universe are not isolated systems. They are often surrounded by matter and embedded in huge electromagnetic fields.
  In General Relativity there are known exact and analytical solutions which describe Schwarzschild or Kerr black holes embedded in the Bonnor-Melvin external electromagnetic field (Ernst 1976) or into the Levi-Civita-Bertotti-Robinson external electromagnetic field (Alekseev-Garcia 1996, Podoslky-Ovcharenko 2025).

- Which are the differences between the two black hole families in external electromagnetic field?
- Which are the differences between the two background electromagnetic fields?

- It is possible to extend and unify these two black hole family to build a black hole embedded into a more general electromagnetic field?

- Theory: General Relativity coupled with Maxwell electromagnetism

$$I[g_{\mu\nu}, A_\mu] := \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[\mathrm{R} - \frac{G}{\mu_0}\,F_{\mu\nu}F^{\mu\nu}\right] \quad .$$

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Schwarzschild in the external Bertotti-Robinson electromagnetic field (Podoslky-Ovcharenko 2025)

$$ds^2 = \frac{1}{\Omega^2}\left[-\Delta_r(r)dt^2 + \frac{dr^2}{\Delta_r(r)} + \frac{r^2 d\theta^2}{1+m^2B^2\cos^2\theta} + r^2\sin^2\theta(1+m^2B^2\cos^2\theta)\Delta_\varphi^2\, d\varphi^2\right]$$

with

$$\Delta_r(r) = \left(1-\frac{2m}{r}-m^2B^2\right)\left(1+B^2r^2\right) \ ,$$

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$$A_\mu = \left(0,0,0,\frac{1+mB^2r\cos^2\theta-\Omega}{B\Omega}\Delta_\varphi\right) \ .$$

For **B=0**: Schwarzschild

For **m=0**: Bertotti-Robinson ($x := \cos\theta$)

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Bertotti-Robinson electromagnetic background

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Thanks to the change of coordinates

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r \to \frac{\sqrt{B^2R^2+\sin^2\theta}}{B\cos\theta} , \qquad x \to \frac{BR}{\sqrt{B^2R^2+\sin^2\theta}} , \qquad B \to \frac{1}{e} ,
$$

it takes a more common form

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ds^2 = -\left(1+\frac{R^2}{e^2}\right)dt^2+\frac{dR^2}{1+\frac{R^2}{e^2}}+e^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right) .
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Schwarzschild in Bertotti-Robinson-Bonnor-Melvin field

Thanks to the Harrison transformation (applied to the Schwarzschild-Bertotti-Robinson seed) we can build Schwarzschild embedded in the Bertotti-Robinson-Bonnor-Melvin (elettro)magnetic field

$$ds^2 = -\bar{f}\,\Delta_\varphi^2 d\varphi^2 + \frac{1}{\bar{f}}\left[\rho^2 dt^2 - e^{2\gamma}\left(\frac{dr^2}{\Delta_r} + \frac{dx^2}{\Delta_x}\right)\right] \ ,$$

with

$$\bar{f}(r,x) = \frac{-4B^4 r^2 \Delta_x}{\left[b(b+2B)\left(B^2 m r x^2 + 1\right) - \left(b^2 + 2bB + 2B^2\right)\Omega\right]^2} \ , \qquad \rho(r,x) = \frac{\sqrt{\Delta_r \Delta_x}}{\Omega^2}$$

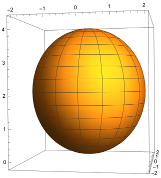
$$\Delta_r(r) \quad := \quad \left(1 - \frac{2m}{r} - m^2 B^2\right)\left(1 + B^2 r^2\right) \ , \qquad \Delta_x(x) := (1-x^2)(1+B^2 m^2 x^2) \,,$$

$$\Omega(r,x) \quad := \quad \sqrt{1 + B^2 r^2 - B^2 r \cos^2\theta (r - 2m - rB^2 m^2)} \ , \qquad \gamma(r,x) := \frac{1}{2}\log\frac{r^4 \Delta_x}{\Omega^4} \,,$$

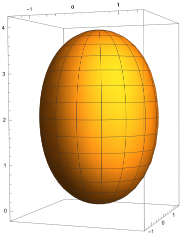
and magnetic potential

$$\bar{A}_\varphi(r,x) = \frac{-2r^2(b+B)\Delta_x}{2\Omega\left(B^2 m r x^2 + 1\right) + b r^2 (b+2B)\Delta_x + 2B^2 r\left[x^2\left(B^2 m^2 r + 2m - r\right) + r\right] + 2} \,.$$

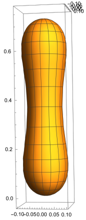
Schwarzschild in Bertotti-Robinson-Bonnor-Melvin field



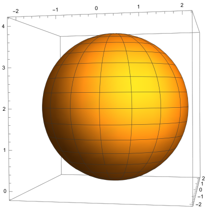
$B = 0.1,\ b = 0.1$



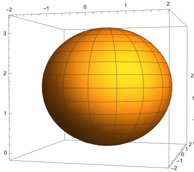
$B = 0.4,\ b = 0.1$



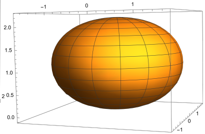
$B = 2,\ b = 0.1$



$B = 0,\ b = 0$



$B = 0.3,\ b = -0.4$



$B = 0.5,\ b = -0.6$

Figure: Embedding in Euclidean three-dimensional space $\mathbb{E}^3$ of the event horizon of the black hole distorted by the presence of both the external Bertotti-Robinson and Bonnor-Melvin magnetic backgrounds, for $m = 1$

Limit to Schwarzschild in Bonnor-Melvin universe

For **B=0**: Schwarzschild embedded in the Bonnor-Melvin (elettro)magnetic field

$$ds^2 = \left[1 + \frac{b^2 r^2}{4}(1-x^2)\right]^2 \left[-\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + \frac{r^2 dx^2}{(1-x^2)}\right] + \frac{r^2(1-x^2)\ d\varphi^2}{\left[1+\frac{b^2 r^2}{4}(1-x^2)\right]^2} \ ,$$

$$A_\mu = \left[0, 0, 0, \frac{2br^2(1-x^2)}{4+b^2r^2(1-x^2)}\right] \ .$$

For **B=0** & **m=0**: Bonnor-Melvin magnetic Universe

$$ds^2 = \left[1 + \frac{b^2 r^2}{4}(1-x^2)\right]^2 \left[-dt^2 + dr^2 + \frac{r^2 dx^2}{(1-x^2)}\right] + \frac{r^2(1-x^2)}{\left[1+\frac{b^2 r^2}{4}(1-x^2)\right]^2} d\varphi^2 \ ,$$

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m=0: Magnetic Bertotti-Robinson-Bonnor-Melvin background

$$ds^2 = \Lambda^2 \left[-(1+B^2r^2)dt^2 + \frac{dr^2}{1+B^2r^2} + \frac{r^2dx^2}{(1-x^2)} \right] + \frac{r^2(1-x^2)}{\Lambda^2 \ \left[1+B^2r^2(1-x^2)\right]^2} d\varphi^2 \ ,$$

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$$\Lambda(r,x) = \frac{-b(b+2B) + (b^2+2bB+2B^2)\sqrt{1+B^2r^2(1-x^2)}}{2B^2[1+B^2r^2(1-x^2)]} \ ,$$

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The general magnetic background coincides with the analytical continuation

$$t \rightarrow i\varphi \ , \qquad \varphi \rightarrow it \ , \qquad Q \rightarrow i(b+B) \ , \qquad k \rightarrow -B^2 \ , \qquad m \rightarrow -\frac{b^2}{2} - bB - B^2 \ ,$$

of the topological Reissner-Nordstrom metric

$$ds^2 = -\left(k - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{k - \frac{2m}{r} + \frac{Q^2}{r^2}} + \frac{dx^2}{1-kx^2} + r^2(1-kx^2)d\varphi^2 \ ,$$

$$A = A_\mu dx^\mu = -\frac{Q}{r}\, dt \ .$$

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$$\Lambda(r,x) = \frac{-b(b+2B)+(b^2+2bB+2B^2)\sqrt{1+B^2r^2(1-x^2)}}{2B^2[1+B^2r^2(1-x^2)]} \ ,$$

$$A_\mu = \left\{ 0, 0, 0, \frac{2(b+B)r^2(1-x^2)}{-(b^2+2bB+2B^2)r^2(1-x^2)+2\left[1+\sqrt{1+B^2r^2(1-x^2)}\right]} \right\} \ .$$

The general magnetic background coincides with the analytical continuation

$$t \to i\varphi \ , \qquad \varphi \to it \ , \qquad Q \to i(b+B) \ , \qquad k \to -B^2 \ , \qquad m \to -\frac{b^2}{2} - bB - B^2 \ ,$$

of the topological Reissner-Nordstrom metric

$$ds^2 = -\left(k - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{k - \frac{2m}{r} + \frac{Q^2}{r^2}} + \frac{dx^2}{1-kx^2} + r^2(1-kx^2)d\varphi^2 \ ,$$

$$A = A_\mu dx^\mu = -\frac{Q}{r} \, dt \ .$$

m=0: Magnetic Bertotti-Robinson-Bonnor-Melvin background

$$ds^2 = \Lambda^2 \left[-(1+B^2r^2)dt^2 + \frac{dr^2}{1+B^2r^2} + \frac{r^2dx^2}{(1-x^2)} \right] + \frac{r^2(1-x^2)}{\Lambda^2 \ \left[1+B^2r^2(1-x^2)\right]^2} d\varphi^2 \ ,$$

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Limit to Schwarzschild in a gravitational external background

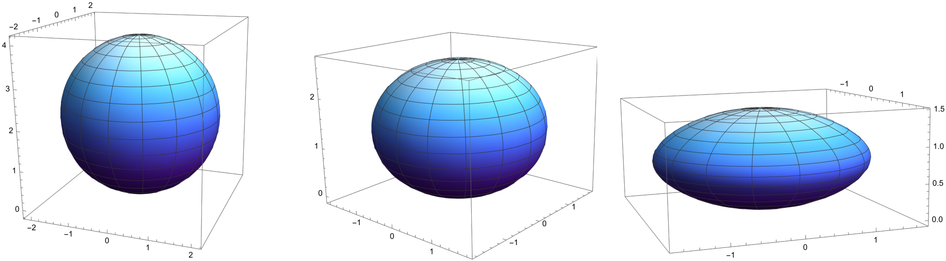
For **b = -B**: Schwarzschild black hole in a gravitational background, $A_\mu = [0,0,0,0]$

$$ds^2 = \frac{(1+B^2 mr\cos^2\theta+\Omega)^2}{4\Omega^4}\left[-f(r)dt^2+\frac{dr^2}{f(r)}+\frac{r^2 d\theta^2}{\Delta_\theta(\theta)}\right]+\frac{4r^2\sin^2\theta\,\Delta_\theta(\theta)\,\Delta_\varphi^2\,d\varphi^2}{(1+B^2 mr\cos^2\theta+\Omega)^2}\,,$$

$$f(r) = \left(1-\frac{2m}{r}-B^2m^2\right)(1+B^2r^2)\,, \qquad \Delta_\theta(\theta) = 1+B^2m^2\cos^2\theta\,,$$

$$\Omega(r,\theta) = \sqrt{1+B^2r[r+(2m+B^2m^2r-r)\cos^2\theta]}\,, \qquad \Delta\varphi = 1/(1+B^2m^2)\,.$$

Static, type I, black hole with a “gravitational hair” and squashed event horizon



(a) $m=1,\ B=0$ **(b)** $m=1,\ B=0.4$ **(c)** $m=1,\ B=0.5$

Pure gravitational background

$$ds^2 = \left[\frac{1+\sqrt{1+B^2 r^2 \sin^2\theta}}{2\left(1+B^2 r^2 \sin^2\theta\right)} \right]^2 \left[-(1+B^2 r^2)dt^2 + \frac{dr^2}{1+B^2 r^2} + r^2 d\theta^2 \right] + \frac{4r^2 \sin^2\theta \ d\varphi^2}{\left(1+\sqrt{1+B^2 r^2 \sin^2\theta}\right)^2}$$

Thanks to the change of coordinates

$$r \to \frac{\sqrt{4p - 4p^2 + B^2 q^2}}{B(2p-1)} \ , \qquad \theta \to \arccos\left(\frac{Bq}{\sqrt{4p-4p^2+B^2q^2}} \right) \ , \qquad \varphi \to \frac{B^2}{2}\varphi \ ,$$

the background is diffeomorphic to

$$ds^2 = p^2 \left[-Q(q)dt^2 + \frac{dq^2}{Q(q)} + \frac{dp^2}{P(p)} \right] + \frac{P(p)}{p^2} d\varphi^2$$

with

$$Q(q) = 1 + B^2 q^2 \ , \qquad P(p) = B^2 p - B^2 p^2 \ .$$

Which corresponds to the double Wick rotation

$$\hat{t} \to i\varphi \ , \qquad \hat{\varphi} \to it \ , \qquad \hat{k} \to -B^2 \ , \qquad \hat{m} \to -\frac{B^2}{2} \ , \qquad \hat{x} \to q \ , \qquad \hat{r} \to p$$

of the hyperbolic Schwarzschild metric

$$d\hat{s}^2 = -\left(\hat{k} - \frac{2\hat{m}}{\hat{r}}\right) d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{k} - \frac{2\hat{m}}{\hat{r}}} + \frac{\hat{r}^2 \, d\hat{x}^2}{1 - \hat{k}\hat{x}^2} + \hat{r}^2 (1 - \hat{k}\hat{x}^2) \, d\hat{\varphi}^2 \ .$$

Pure gravitational background

$$ds^2 = \left[\frac{1+\sqrt{1+B^2r^2\sin^2\theta}}{2\left(1+B^2r^2\sin^2\theta\right)} \right]^2 \left[-(1+B^2r^2)dt^2 + \frac{dr^2}{1+B^2r^2} + r^2 d\theta^2 \right] + \frac{4r^2\sin^2\theta \ d\varphi^2}{\left(1+\sqrt{1+B^2r^2\sin^2\theta}\right)^2}$$

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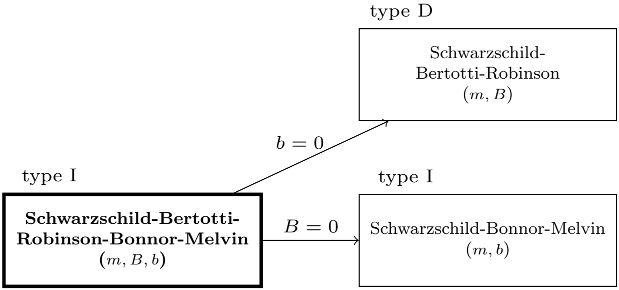
Summary & notable limits

Specializations of the electrovacuum solutions presented this talk



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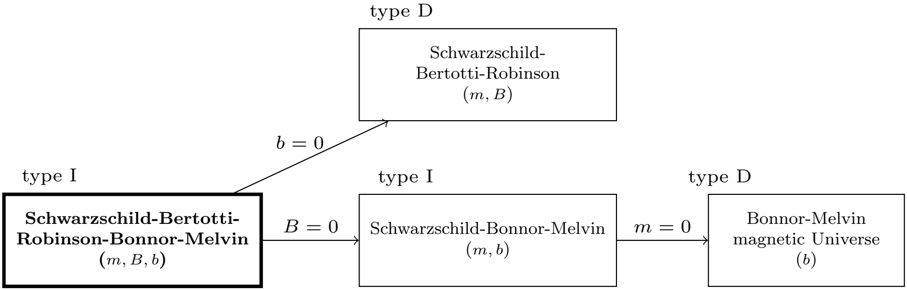

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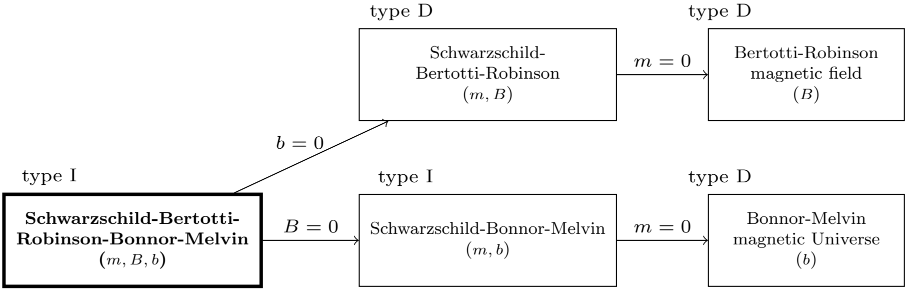

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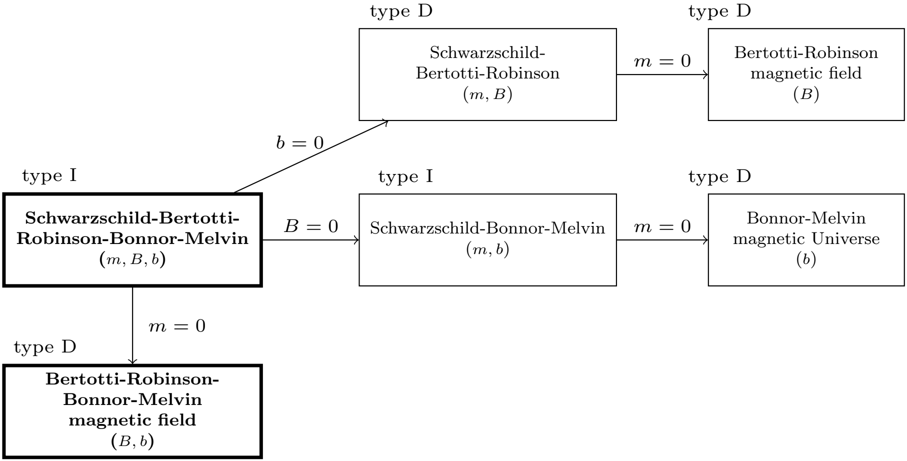

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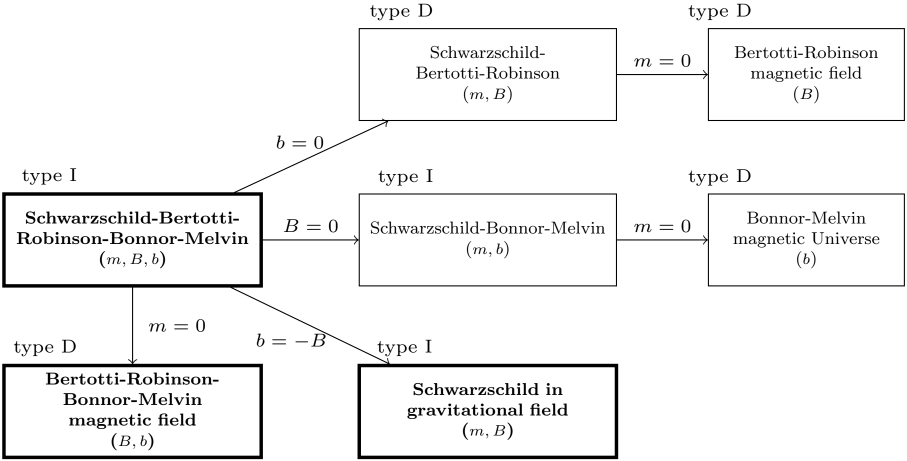

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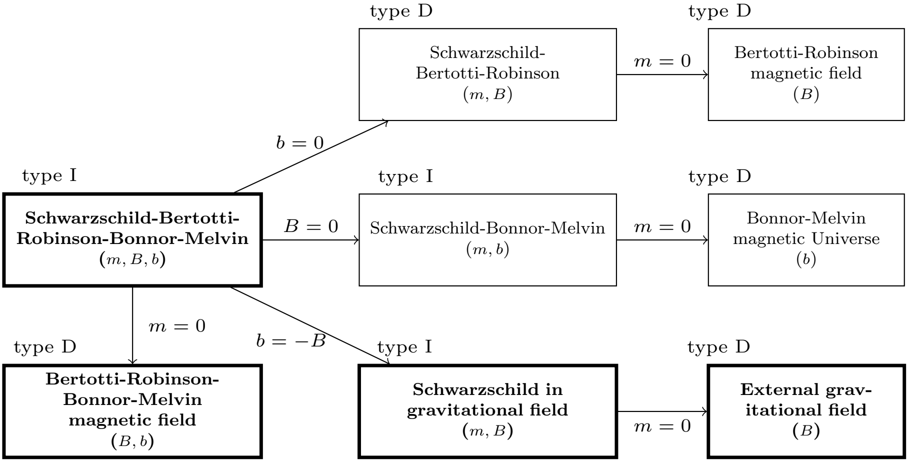

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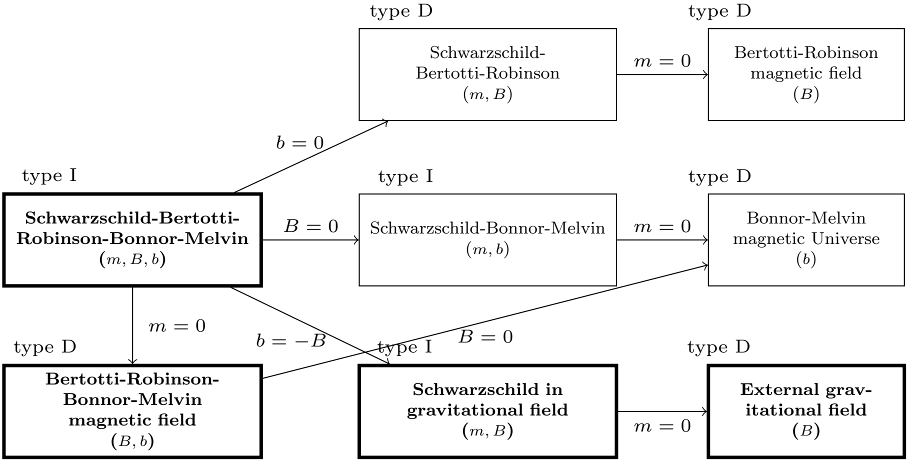

That's all

¡Thank You!

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PRD 112 (2025) 10, 104077

Theory under consideration: General Relativity coupled with Maxwell electromagnetism

$$I[g_{\mu\nu}, A_\mu] := \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[\mathrm{R} - \frac{G}{\mu_0} \; F_{\mu\nu} F^{\mu\nu} \right] \quad .$$

Field equations for the metric $g_{\mu\nu}$ and electromagnetic vector potential A_μ

$$\mathrm{R}_{\mu\nu} - \frac{\mathrm{R}}{2} g_{\mu\nu} = \frac{2G}{\mu_0} \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \quad ,$$

$$\partial_\mu (\sqrt{-g} F^{\mu\nu}) = 0 \quad .$$

The Faraday tensor $F_{\mu\nu}$ is defined from the gauge potential, $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$.

The most generic axisymmetric and stationary spacetime, containing two commuting Killing vectors ∂_t and ∂_φ, can be written, for this theory, in the Lewis-Weyl-Papapetrou (LWP) form as

$$ds^2 = -f \left(dt - \omega d\varphi \right)^2 + f^{-1} \left[\rho^2 d\varphi^2 + e^{2\gamma} \left(d\rho^2 + dz^2 \right) \right] \; .$$

All the three structure functions appearing in the metric f, ω and γ depends only on the non-Killing coordinates (ρ, z).

A generic electromagnetic potential compatible with the spacetime symmetries, and the circularity of the LWP metric, is given by $A = A_t(\rho, z) dt + A_\varphi(\rho, z) d\varphi$.

Theory under consideration: General Relativity coupled with Maxwell electromagnetism

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Introduction: Theory and Field Equations

Theory under consideration: General Relativity coupled with Maxwell electromagnetism

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Ernst (Phys. Rev. 1968) discovered that, when the Einstein field equations are restricted to the axisymmetric and stationary LWP ansatz, they reduce to a couple of complex vectorial differential equations

Ernst Equations

$$\begin{aligned}\left(\mathrm{Re}\ \mathcal{E} + |\mathbf{\Phi}|^2\right) \nabla^2 \mathcal{E} &= \left(\vec{\nabla}\mathcal{E} + 2\ \mathbf{\Phi}^* \vec{\nabla}\mathbf{\Phi}\right) \cdot \vec{\nabla}\mathcal{E} \quad , \\ \left(\mathrm{Re}\ \mathcal{E} + |\mathbf{\Phi}|^2\right) \nabla^2 \mathbf{\Phi} &= \left(\vec{\nabla}\mathcal{E} + 2\ \mathbf{\Phi}^* \vec{\nabla}\mathbf{\Phi}\right) \cdot \vec{\nabla}\mathbf{\Phi} \quad .\end{aligned}$$

The complex Ernst potential are defined as

$$\mathbf{\Phi} := A_t + i\tilde{A}_\varphi \qquad , \qquad \mathcal{E} := f - |\mathbf{\Phi}\mathbf{\Phi}^*| + ih \ ,$$

where $\tilde{A}_\varphi$ and h can be obtained from

$$\begin{aligned}\vec{\nabla}\tilde{A}_\varphi &:= -fr^{-1}\vec{e}_\varphi \times (\vec{\nabla}A_\varphi - \omega\vec{\nabla}A_t) \ , \\ \vec{\nabla}h &:= -f^2r^{-1}\vec{e}_\varphi \times \vec{\nabla}\omega - 2\ \mathrm{Im}(\mathbf{\Phi}^*\vec{\nabla}\mathbf{\Phi}) \ .\end{aligned}$$

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Ernst equations can be derived by an effective action

$$I(\mathcal{E},\mathbf{\Phi}) = \int dz \int d\rho \left[\frac{\left(\vec{\nabla}\mathcal{E} + 2\mathbf{\Phi}^* \vec{\nabla}\mathbf{\Phi}\right)\left(\vec{\nabla}\mathcal{E}^* + 2\mathbf{\Phi}\vec{\nabla}\mathbf{\Phi}^*\right)}{\left(\mathcal{E} + \mathcal{E}^* + 2\mathbf{\Phi}\mathbf{\Phi}^*\right)^2} - \frac{\vec{\nabla}\mathbf{\Phi}\vec{\nabla}\mathbf{\Phi}^*}{\mathcal{E} + \mathcal{E}^* + 2\mathbf{\Phi}\mathbf{\Phi}^*} \right]$$

This action has a set of Lie point symmetries which form the $SU(2,1)$ group. These symmetries can be written as a set of five independent transformation:

Ernst Equations Symmetries (Lie point)

$$\begin{array}{lll}
(I) & \mathcal{E} \longrightarrow \mathcal{E}' = \lambda\lambda^*\mathcal{E} \quad , & \mathbf{\Phi} \longrightarrow \mathbf{\Phi}' = \lambda\mathbf{\Phi} \quad , \\
(II) & \mathcal{E} \longrightarrow \mathcal{E}' = \mathcal{E} + i\, b \quad , & \mathbf{\Phi} \longrightarrow \mathbf{\Phi}' = \mathbf{\Phi} \quad , \\
(III) & \mathcal{E} \longrightarrow \mathcal{E}' = \dfrac{\mathcal{E}}{1 + ic\mathcal{E}} \quad , & \mathbf{\Phi} \longrightarrow \mathbf{\Phi}' = \dfrac{\mathbf{\Phi}}{1 + ic\mathcal{E}} \quad , \\
(IV) & \mathcal{E} \longrightarrow \mathcal{E}' = \mathcal{E} - 2\beta^*\mathbf{\Phi} - \beta\beta^* \quad , & \mathbf{\Phi} \longrightarrow \mathbf{\Phi}' = \mathbf{\Phi} + \beta \quad , \\
(V) & \mathcal{E} \longrightarrow \mathcal{E}' = \dfrac{\mathcal{E}}{1 - 2\alpha^*\mathbf{\Phi} - \alpha\alpha^*\mathcal{E}} \quad , & \mathbf{\Phi} \longrightarrow \mathbf{\Phi}' = \dfrac{\mathbf{\Phi} + \alpha\mathcal{E}}{1 - 2\alpha^*\mathbf{\Phi} - \alpha\alpha^*\mathcal{E}}
\end{array}$$

where $b, c \in \mathbb{R}$ and $\alpha, \lambda, \beta \in \mathbb{C}$.

Some of these transformation are just gauge symmetries and can be reabsorbed by a coordinate transformation, while others actually have non-trivial physical effects.

Introduction: Symmetries of Ernst Equations

Ernst equations can be derived by an effective action

$$I(\mathcal{E},\mathbf{\Phi}) = \int dz \int d\rho \left[\frac{(\vec{\nabla}\mathcal{E} + 2\mathbf{\Phi}^*\vec{\nabla}\mathbf{\Phi})(\vec{\nabla}\mathcal{E}^* + 2\mathbf{\Phi}\vec{\nabla}\mathbf{\Phi}^*)}{(\mathcal{E} + \mathcal{E}^* + 2\mathbf{\Phi}\mathbf{\Phi}^*)^2} - \frac{\vec{\nabla}\mathbf{\Phi}\vec{\nabla}\mathbf{\Phi}^*}{\mathcal{E} + \mathcal{E}^* + 2\mathbf{\Phi}\mathbf{\Phi}^*} \right]$$

This action has a set of Lie point symmetries which form the $SU(2,1)$ group. These symmetries can be written as a set of five independent transformation:

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