General approach to BSW effect: nonequatorial particle motion and presence of force

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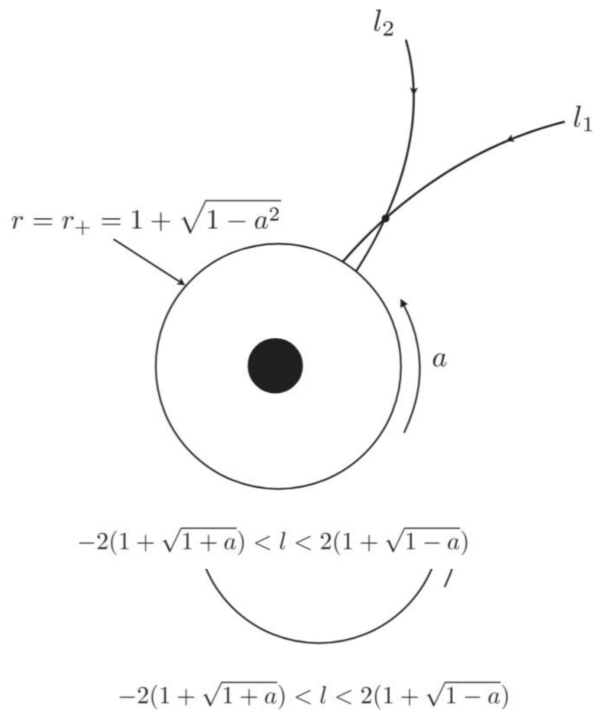
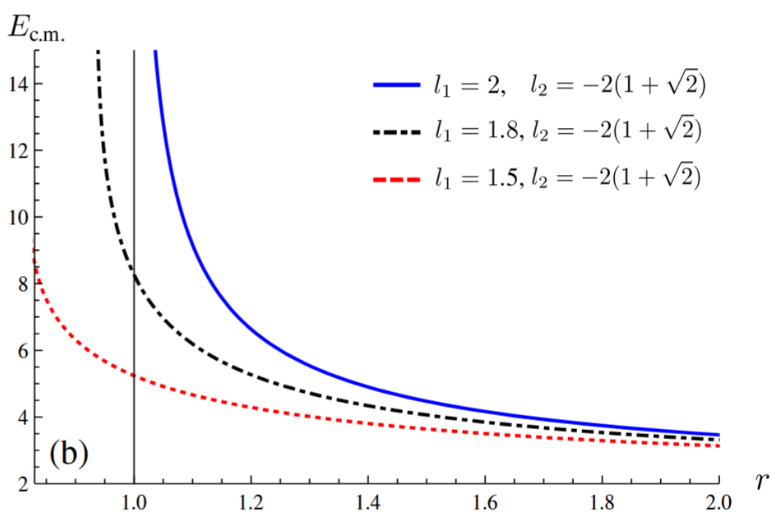
Two particles falling towards horizon

For some special value of angular momentum, energy in the center-of-mass frame diverges near ***extremal*** horizon





M. Banados, J. Silk and S. M. West, Kerr Black Holes as Particle Accelerators to Arbitrarily High Energy, Phys. Rev. Lett. ***103****, 111102*

What is crucial for generalization?

To understand whether it is a general property of black holes, let us consider generic axially symmetric spacetime

$$ds^2 = -N^2 dt^2 + g_{\varphi\varphi}(dt - \omega d\varphi)^2 + \frac{dr^2}{A} + g_{\theta\theta} d\theta^2.$$

4-velocity of the infalling particle in the equatorial plane is given by

$$u^\mu = \left(\frac{\mathcal{X}}{N^2}, \frac{\mathcal{L}}{g_\varphi} + \frac{\omega \mathcal{X}}{N^2}, \sigma \frac{\sqrt{A}}{N} P, 0\right).$$

where $\mathcal{X} = \varepsilon - \omega \mathcal{L}$,

$$P = \sqrt{\mathcal{X}^2 - N^2\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)},$$

Gamma-factor of relative motion of 2 particles is given by

$$\gamma = \frac{\mathcal{X}_1 \mathcal{X}_2 - P_1 P_2}{N^2} - \frac{\mathcal{L}_1 \mathcal{L}_2}{g_{\varphi\varphi}}.$$

It becomes divergent only if on the horizon X_1=0 or X_2=0 (such particles are called ***critical***)

$$E^2_{c.m.} = m^2_1 + m^2_2 + 2m_1 m_2 \gamma \qquad \gamma = -u_{1\mu} u_2^{\mu}$$

generalization?

BSW effect holds for any axially symmetric spacetimes

This may happen for charged particles and non-rotating spacetimes

Analogous phenomenon is observed when a black hole is non-extremal (but particle has to be nearly-critical).

What happens with the presence of a generic force? For other types of horizons?

Types of horizon: nonextremal, extremal (double), ultraextremal (triple, etc.)

Setup for calculations

- We assume that the spacetime is axially symmetric

$$ds^2 = -N^2 dt^2 + g_\phi (d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_\theta d\theta^2$$

- Moreover, particle is not in a free motion, thus its energy and angular momentum ***are not*** constant and are assumed to be generic functions of radial coordinate

- As we are interested in the near-horizon collision, we assume that the metric functions have such an expansion

$$N^2 = \kappa_p v^p + o(v^p), \quad A = A_q v^q + o(v^q),$$

$$\omega = \omega_H + \hat{\omega}_k v^k + \ldots + \hat{\omega}_{l-1} v^{l-1} + \omega_l(\theta) v^l + o(v^l),$$

$$g_a = g_{aH} + g_{a1} v + o(v).$$

Horizon is regular if

$$k \geq \left[\frac{p-q+3}{2}\right], \quad l \geq p,$$

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What we aim to find

Central aim of several our works was to find:
For what types of particles and for what horizons does the high-energy collision effect survive while the forces, acting on such particles, remain finite?

- 1-st case: equatorial motion

In this case, the 4 velocity is given by

$$u^{\mu} = \left(\frac{\mathcal{X}}{N^2}, \frac{\mathcal{L}}{g_{\varphi}} + \frac{\omega \mathcal{X}}{N^2}, \sigma \frac{\sqrt{A}}{N} P, 0 \right).$$

Now, X is not a constant and is defined by the external force

$$\mathcal{X} = \varepsilon - \omega \mathcal{L}, \qquad P = \sqrt{\mathcal{X}^2 - N^2 \left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}} \right)},$$

As it is not a constant, we assume that near the horizon X behaves as
$\mathcal{X} \approx \mathcal{X}_s v^s,$

Types of particles

Near-horizon limit: $N \to 0, A \to 0$

$$u^r \approx \frac{\sqrt{A}}{N}$$ usual

If $\left|u^r\right|$ changes slower than $\sqrt{A}$ but faster than $\frac{\sqrt{A}}{N}$ subcritical

if $\left|u^r\right| \sim \sqrt{A}$ critical

if faster than $\sqrt{A}$ ultracritical

TABLE I. Characteristics of the near-horizon behavior of u^r, u^t, $\mathcal{X}$, and the proper time τ. Here, the proper time changes as $\tau \sim v^{-\alpha}$. The value $\alpha = 0$ means that the proper time logarithmically diverges $\tau \sim |\ln v|$.

	Type	c	β	s	$\alpha = c - 1$
1	Usual	$\frac{q-p}{2}$	p	0	$\frac{q-p-2}{2}$
2	Subcritical	$\frac{q-p}{2} < c < \frac{q}{2}$	$\frac{p+q}{2} - c$	$\frac{p-q}{2} + c,\ 0 < s < \frac{p}{2}$	$\frac{q-p-2}{2} < \alpha < \frac{q-2}{2}$
3	Critical	$\frac{q}{2}$	$\frac{p}{2}$	$\frac{p}{2}$	$\frac{q-2}{2}$
4	Ultracritical	$c > \frac{q}{2}$	$\frac{p}{2}$	$\frac{p}{2}$	$\alpha > \frac{q-2}{2}$

BSW effect and types of particles

By analysing the gamma-factor for various types of particles, we obtained such table of possible scenarios

$$\gamma = \frac{\mathcal{X}_1\mathcal{X}_2 - P_1P_2}{N^2} - \frac{\mathcal{L}_1\mathcal{L}_2}{g_{\varphi\varphi}}.$$

TABLE II. The possibility of BSW phenomenon for different types of particles. D means that the gamma factor diverges, and R means that it is regular.

	First particle	Second particle	d	γ
1	Usual	Usual	0	R
2	Usual	Subcritical	s_2	D
3	Subcritical	Subcritical	$\lvert s_1 - s_2\rvert$	D if $s_1 \neq s_2$ R if $s_1 = s_2$
4	Usual or subcritical	Critical or ultracritical	$p/2 - s_1$	D
5	Critical or ultracritical	Critical or ultracritical	0	R

Thus, generally, BSW is possible if one particle is non-usual (while second is usual or subcritical)

What happens with the presence of a generic force? For other types of horizons?

Results for the finite force

Non-regular horizon

$$0 < k < \frac{p-q}{2} + 1, \quad \text{region I,}$$

$$\frac{p-q}{2} + 1 \le k < \frac{p+1-q/2}{2}, \quad \text{region II,}$$

$$\frac{p+1-q/2}{2} \le k < \frac{p}{2}, \quad \text{region III,}$$

$$k \ge p/2, \quad \text{region IV.}$$

Diverging force

TABLE IV. Classification of near-horizon trajectories for different k regions for $q > 2$ (ultraextremal horizon). The fourth solution in (64) is not presented in this table. Definitions of different k regions are given in Fig. 1.

	k Region	n_1 Range	c	Type of trajectory
		Stationary metric		
1	I	For any type of trajectory, n_1 is negative (forces diverge)		
2	II	$2k + \frac{q-2-2p}{2} < n_1 < \frac{q-p}{2} - 1 + k$	$n_1 + 1 + p/2 - k$	Subcritical
		$n_1 = \frac{q-p}{2} - 1 + k$	$q/2$	Critical
		$n_1 = \frac{q-p}{2} - 1 + k$ and (28)	Any $c > \frac{q}{2}$	Ultracritical
3	III	$\max(0, \frac{q-2-2p}{2}) < n_1 \le 2k + \frac{q-2-2p}{2}$	$\frac{2n_1+2+q}{4}$	Subcritical
		$2k + \frac{q-2-2p}{2} < n_1 < \frac{q-p}{2} - 1 + k$	$n_1 + 1 + p/2 - k$	Subcritical
		$n_1 = \frac{q-p}{2} - 1 + k$	$q/2$	Critical
		$n_1 = \frac{q-p}{2} - 1 + k$ and (28)	Any $c > \frac{q}{2}$	Ultracritical
4	IV	$\max(0, \frac{q-2-2p}{2}) < n_1 < \frac{q-2}{2}$	$\frac{2n_1+2+q}{4}$	Subcritical
		$n_1 = \frac{q-2}{2}$	$q/2$	Critical
		$n_1 = \frac{q-2}{2}$ and (28)	Any $c > \frac{q}{2}$	Ultracritical
		Static metric		
5	$k = 0$	Same results as in IV for stationary metric		

TABLE VI. Classification of near-horizon trajectories for different k regions for $q < 2$ (nonextremal horizon). Regions II and III in this case are absent, so they are not presented in this table. The fourth solution in (64) is also not presented in this table. Definitions of different k regions are given in Fig. 1.

	k Region	n_1 Range	c	Type of trajectory
		Stationary metric		
1	I and IV	For any type of trajectory, n_1 is negative (forces diverge)		
		Static metric		
2	$k = 0$	For any type of trajectory, n_1 is negative (forces diverge)		

TABLE V. Classification of near-horizon trajectories for different k regions for $q = 2$ (extemal horizon). Regions II and III in this case are absent, so they are not presented in this table. The fourth solution in (64) is also not presented in this table. Definitions of different k regions are given in Fig. 1.

	k Region	n_1 Range	c	Type of trajectory
		Stationary metric		
1	I	For any type of trajectory, n_1 is negative (forces diverge)		
2	IV	$n_1 = 0$	1	Critical
		$n_1 = 0$ and (28)	Any $c > 1$	Ultracritical
		Static metric		
3	$k = 0$	Same results as in IV for stationary metric		

Results for the finite force

TABLE VIII. Classification of cases when forces are finite for different types of horizons and trajectories.

	Type of horizon	Type of trajectory	Region of k
1	Nonextremal	All types	Absent
2	Extremal	Subcritical	Absent
		Critical	$k \geq p/2$ or $k = 0$
		Ultracritical	
3	Ultraextremal	Subcritical	$k \geq \frac{p-q}{2} + 1$ or $k = 0$
		Critical	
		Ultracritical	

Force remains finite only for extremal and ultraextremal horizons

However, there is a serious drawback: **usually, for critical particles the proper time diverges**

We were able to prove that *for any non-usual particle with the finite proper time either the force is infinite, or the horizon is non regular*.

Thus, for all the cases when the BSW effect is possible, proper time of reaching the horizon diverges (***kinematic censorship***). Is there any way that allows bypassing this issue?

Kinematic censorship

In any event only finite energy can be released.
It can be unbounded but it cannot be literally infinite.

Nonequatorial motion

The scenarios of high energy particle collisions, found previously in the context of the BSW effect for equatorial motion, do not change qualitatively in the nonequatorial case

General approach

Near-fine-tuned particles

Near-fine-tuned particles

Original BSW: extremal BH horizon

Nonextremal BH: critical particle cannot reach horizon Is BSW possible?

Yes! To this end, has to consider near-fine-tuned particle

$$\mathcal{X} = \delta + X_s v^s + o(v^s), \quad s > 0,$$

where δ is some small parameter

Thus we have two small parameters: and the point of collision v_c

Near-fine-tuned particles

Thus we have two small parameters: δ and the point of collision v_c

The case $\delta \ll v_c$ effectively corresponds to subcritical, critical and ultracritical particles

$$\mathcal{X} = \delta + X_s v^s + o(v^s),$$

$\delta \to 0$

The case $\delta \gg v_c$ effectively corresponds to usual particles

$$\mathcal{X} = \delta + X_s v^s + o(v^s),$$

$X_s v^s \to 0$

The case δ of order of v_c gives new results

$$\mathcal{X} = \delta + X_s v^s + o(v^s),$$

δ and $X_s v^s$: Of the same order

Classification of particles

TABLE I. Table showing classification of different near-fine-tuned particles.

Condition	Type of particle with nonzero $\delta \ll 1$	Abbreviation
$s < p/2$	Near subcritical	NSC
$s = p/2$	Near critical	NC
$s = p/2$ and (24)	Near ultracritical	NUC
$s > p/2$	Near overcritical	NOC

Thus, we have two small parameters: δ and the point of collision v_c

Near subcritical (NSC) - *X* tends to δ, but with the rate, smaller than *N*

Near critical (NC) - *X* tends to δ with the same rate as *N*

Near ultracritical (NUC) - *X* tends to δ with the same rate as *N*, but several first terms in the expansion of *P* cancel

Near overcritical (NOC) - *X* tends to δ with the rate, higher than *N*.

$$P = \sqrt{\mathcal{X}^2 - N^2\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)},$$

Gamma-factor

TABLE II. Table showing behavior of γ factor for $v_c \sim v_{e,t}$ in a case when first particle is fine-tuned and second is near-fine-tuned. Here d is defined by relation $\gamma \sim v_c^{-d}$.

	First particle	Second particle	BSW is possible
1	U or SC	NSC	$\|s_1 - s_2\|$ or $\|s_1 - r_2\|$ if $\delta_2 = -X_s^{(2)} v_c^s + B_r^{(2)} v_c^r$
2	C or UC	NSC	$\frac{p}{2} - s_2$ or $\frac{p}{2} - r_2$ if $\delta_2 = -X_s^{(2)} v_c^s + B_r^{(2)} v_c^r$
3	U or SC	NC, NUC, or NOC	$\frac{p}{2} - s_1$
4	C or UC	NC, NUC, or NOC	0

BSW is impossible

TABLE III. Table showing behavior of γ factor for $v_c \sim v_{e,t}$ in a case when both particles are near-fine-tuned. Here d is defined by relation $\gamma \sim v_c^{-d}$.

	First particle	Second particle	BSW is possible
1	NSC	NSC	$\|s_1 - s_2\|$ $\|r_1 - s_2\|$ if $\delta_1 = -X_s^{(1)} v_c^s + B_r^{(1)} v_c^r$ $\|s_1 - r_2\|$ if $\delta_2 = -X_s^{(2)} v_c^s + B_r^{(2)} v_c^r$ $\|r_1 - r_2\|$ if $\delta_{1,2} = -X_s^{(1,2)} v_c^s + B_r^{(1,2)} v_c^r$
2	NSC	NC, NUC, or NOC	$\frac{p}{2} - s_1$
3	NC, NUC, or NOC	NC, NUC, or NOC	0

BSW is impossible

Finiteness of force

TABLE IV. Classification of near-horizon trajectories for different k regions for $q > 2$ (ultraextremal horizon). The fourth solution in (108) is not presented in this table.

	k region	n_1 range	s	Type of trajectory
		Stationary metric		
1	I	For any type of trajectory n_1 is negative (forces diverge)		
2	II and III	$\max(0, \frac{q-p}{2} - 1) < n_1 \leq k + \frac{q-p}{2} - 1$	First and second in (108)	NSC
		$n_1 = k + \frac{q-p}{2} - 1$	Second in (108)	NC and NOC
		$n_1 = k + \frac{q-p}{2} - 1$ and (24)	Second in (108)	NUC
3	IV	$\max(0, \frac{q-p}{2} - 1) < n_1 < \frac{q-2}{2}$	First in (108)	NSC
		$n_1 = \frac{q-2}{2}$	First in (108)	NC
		$n_1 = \frac{q-2}{2}$ and (24)	First in (108)	NUC
		$\frac{q-2}{2} < n_1 \leq k + \frac{q-p}{2} - 1$	First and second in (108)	NOC
		Static metric		
4	$k = 0$	Same results as in IV for stationary metric		NSC, NC, and NUC
		$n_1 > \frac{q-2}{2}$	Third in (108)	NOC

TABLE V. Classification of near-horizon trajectories for different k regions for $q = 2$ (ultraextremal horizon). The fourth solution in (108) is not presented in this table.

	k region	n_1 range	s	Type of trajectory
		Stationary metric		
1	I	For any type of trajectory n_1 is negative (forces diverge)		
2	IV	$n_1 = 0$	First in (108)	NC
		$n_1 = 0$ and (24)	First in (108)	NUC
		$0 \leq n_1 \leq k + \frac{q-p}{2} - 1$	First and second in (108)	NOC

TABLE VI. Classification of near-horizon trajectories for different k regions for $q < 2$ (ultraextremal horizon). The fourth solution in (108) is not presented in this table.

	k region	n_1 range	s	Type of trajectory
		Stationary metric		
1	I	For any type of trajectory n_1 is negative (forces diverge)		
2	IV	$0 < n_1 \leq k + \frac{q-p}{2} - 1$	First and second in (108)	NOC
		Static metric		
3	$k = 0$	$0 < n_1$	Third in (108)	NOC

Non-regular horizon

$$0 < k < \frac{p-q}{2} + 1, \quad \text{region I},$$

$$\frac{p-q}{2} + 1 \leq k < \frac{p+1-q/2}{2}, \quad \text{region II},$$

$$\frac{p+1-q/2}{2} \leq k < \frac{p}{2}, \quad \text{region III},$$

$$k \geq p/2, \quad \text{region IV}.$$

Diverging force

Finite force

This part was absent for fine-tuned particles!

Finiteness of force

TABLE VII. Classification of cases when forces are finite for different types of horizons and trajectories.

	Type of horizon	Type of trajectory	Region of k
1	Nonextremal	NOC	IV or static
2	Extremal	NC, NUC, and NOC	IV or static
3	Ultraextremal	NSC, NC, NUC, and NOC	II, III, IV, or static

May achieve infinity

Cannot achieve infinity

Also, for such particles the proper time is large, but finite!

Finiteness of force

Type of horizon	1-st particle's type and range			2-nd particles's type and range			
Extremal	NC, NUC, NOC	$v_c \ll v_{e,t}^{(1)}$	$\delta_1 \gg v_c^{p/2}$	NC, NUC	$v_c \gg v_{e,t}^{(2)}$	$\delta_2 \ll v_c^{p/2}$	4
	U						
	U			C, UC			
Ultraextremal	NSC, NC, NUC, NOC	$v_c \ll v_{e,t}^{(1)}$	$\delta_1 \gg v_c^{p/2}$	NSC	$v_c \gg v_{e,t}^{(2)}$	$\delta_2 \ll v_c^{s}$	2
	U						
	U			SC			
	NSC, NC, NUC, NOC	$v_c \gg v_{e,t}^{(1)}$	$\delta_1 \ll v_c^{p/2}$	NSC	$v_c \gg v_{e,t}^{(2)}$	$\delta_2 \ll v_c^{s}$	3
	SC						
	SC			SC			
	NSC	$v_c \ll v_{e,t}^{(1)}$	$\delta_1 \gg v_c^{s}$	NC, NUC	$v_c \gg v_{e,t}^{(2)}$	$\delta_2 \ll v_c^{p/2}$	4
	U						
	U			C, UC			
	NSC	$v_c \gg v_{e,t}^{(1)}$	$\delta_1 \ll v_c^{s}$	NC, NUC	$v_c \gg v_{e,t}^{(2)}$	$\delta_2 \ll v_c^{p/2}$	4
	SC						
	SC			C, UC			

Original BSW

Finiteness of force

Type of horizon	1-st particle's type and range			2-nd particle's type and range			
Non-extremal	NOC	$v_c \ll v_t^{(1)}$	$\delta_1 \gg v_c^{p/2}$	NOC	$v_c \sim v_t^{(2)}$	$\delta_2 \sim v_c^{p/2}$	(89)
	U						
Extremal	NC, NUC, NOC	$v_c \ll v_{e,t}^{(1)}$	$\delta_1 \gg v_c^{p/2}$	NC, NUC, NOC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{p/2}$	(89)
	U						
Ultraextremal	NSC	$v_c \ll v_{e,t}^{(1)}$	$\delta_1 \gg v_c^{s}$	NSC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{s}$	(83)
	U						
	NSC	$v_c \sim v_{e,t}^{(1)}$	$\delta_1 \sim v_c^{s}$	NSC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{s}$	(91)
	NSC	$v_c \gg v_{e,t}^{(1)}$	$\delta_1 \ll v_c^{s}$	NSC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{s}$	(83)
	SC						
	NSC	$v_c \ll v_{e,t}^{(1)}$	$\delta_1 \gg v_c^{s}$	NC, NUC, NOC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{p/2}$	(89)
	U						
	NSC	$v_c \sim v_{e,t}^{(1)}$	$\delta_1 \sim v_c^{s}$	NC, NUC, NOC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{p/2}$	(92)
	NSC	$v_c \sim v_{e,t}^{(1)}$	$\delta_1 \sim v_c^{s}$	NC, NUC	$v_c \gg v_{e,t}^{(2)}$	$\delta_2 \ll v_c^{p/2}$	(85)
				C, UC			
	NSC	$v_c \gg v_{e,t}^{(1)}$	$\delta_1 \ll v_c^{s}$	NC, NUC, NOC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{p/2}$	(89)
	SC						
	NC, NUC, NOC	$v_c \ll v_{e,t}^{(1)}$	$\delta_1 \gg v_c^{p/2}$	NC, NUC, NOC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{p/2}$	(89)
	U						

Wide variety of various cases!

Summary

Possible set of high energy scenarios with finite force depending on type of horizons and trajectories of colliding particles

Validity of kinematic censorship

Thank you for your attention!