Energy Conditions

in Non–Minimally Coupled Weyl Connection Gravity

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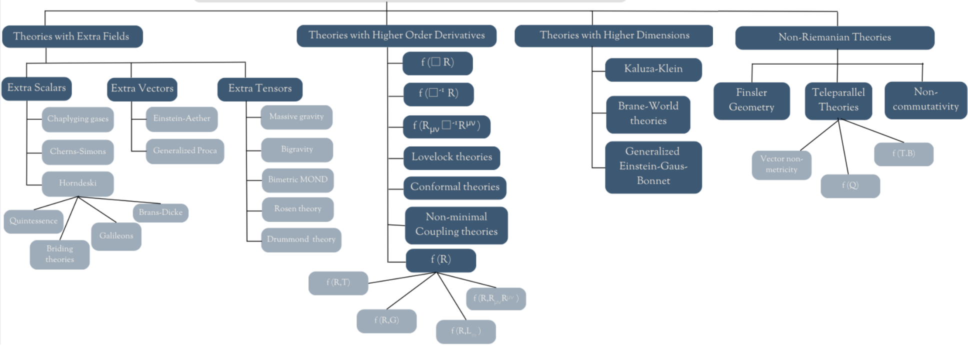
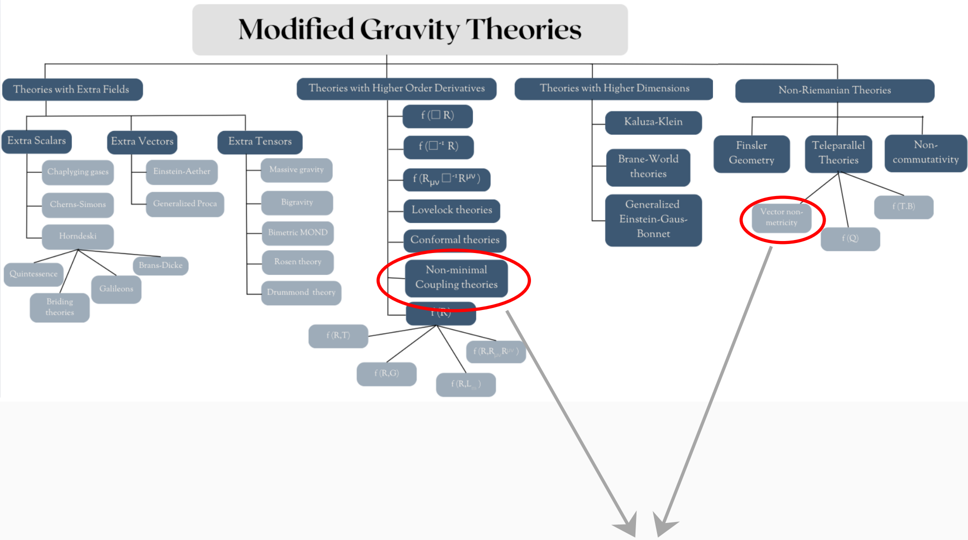
Table of contents

Modified Gravity Theories





Non–Minimally Coupled Weyl Connection Gravity

The Weyl connection introduces a vector field that provides non-metricity properties:

$$D_\lambda g_{\mu\nu} = A_\lambda g_{\mu\nu},$$

where A_λ is the Weyl vector field and $D_\lambda g_{\mu\nu} = \nabla_\lambda g_{\mu\nu} - \bar{\bar{\Gamma}}^\rho_{\mu\lambda} g_{\rho\nu} - \bar{\bar{\Gamma}}^\rho_{\nu\lambda} g_{\rho\mu}$.

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The **generalized Ricci tensor** is given by:

$$\bar{R}_{\mu\nu} = R_{\mu\nu} + \underbrace{\frac{1}{2}A_\mu A_\nu + \frac{1}{2}g_{\mu\nu}\left(\nabla_\lambda - A_\lambda\right)A^\lambda + \tilde{F}_{\mu\nu} + \frac{1}{2}\left(\nabla_\mu A_\nu + \nabla_\nu A_\mu\right)}_{\bar{\bar{R}}_{\mu\nu}},$$

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The **scalar curvature** is given by:

$$\bar{R} = R + \underbrace{3\nabla_\lambda A^\lambda - \frac{3}{2}A_\lambda A^\lambda}_{\bar{\bar{R}}}.$$

Non-minimal matter–curvature coupling model, with Weyl connection, considering action functional:

$$S = \int \left(f_1(\bar{R}) + f_2(\bar{R})\mathcal{L}\right) \sqrt{-g} d^4x,$$

where $f_1(\bar{R})$ and $f_2(\bar{R})$ are generic functions of the scalar curvature.

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Varying the action with respect to the vector field, we obtain the **constraint-like equations**:

$$\nabla_\lambda \Theta(\bar{R}) = -A_\lambda \Theta(\bar{R}),$$

where $\Theta(\bar{R}) = F_1(\bar{R}) + F_2(\bar{R})\mathcal{L}$ and $F_i(\bar{R}) = \frac{df_i(\bar{R})}{d\bar{R}}$, $i \in \{1, 2\}$.

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Varying the action with respect to the metric, we obtain the **field equations**:

$$\bar{R}_{(\mu\nu)}\Theta(\bar{R}) - \frac{1}{2} g_{\mu\nu} f_1(\bar{R}) = \frac{f_2(\bar{R})}{2} T_{\mu\nu}.$$

It is possible to derive a **non-conservation law for energy–momentum tensor**:

$$\nabla_\mu T^{\mu\nu} = \frac{2}{f_2(\bar{R})}\left[\frac{F_2(\bar{R})}{2}(g^{\mu\nu}\mathcal{L} - T^{\mu\nu})\nabla_\mu R + \nabla_\mu(\Theta(\bar{R})B^{\mu\nu}) - \frac{1}{2}(F_1(\bar{R})g^{\mu\nu} + F_2(\bar{R})T^{\mu\nu})\nabla_\mu \bar{\bar{R}}\right],$$

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- A generalization of the coupling can result in an extra force in the geodesic equation;
- Non-metricity also plays a significant role, introducing further contributions to the exchange between geometry and matter sectors.

General Relativity:

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2 - R_{\mu\nu}u^\mu u^\nu,$$

where the quadratic invariants of the shear and vorticity tensors are given, respectively, by $\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}/2$ and $\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}/2$.

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Schematically:

$$\dot{\theta} = \Big(\textbf{GR terms}\Big) + \Big(\textbf{non-metricity contributions}\Big)$$

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The Case of Pure Weyl Non-Metricity:

$$\dot{\bar{\theta}} = \frac{1}{2}\bar{\theta}^2 - 2\sigma^2 + 2\omega^2 - \bar{R}_{\mu\nu}u^\mu u^\nu,$$

[Iosifidis, Tsagas & Petkou (2018)]

where $\bar{\theta} = \left(\theta - 2\frac{\dot{\ell}}{\ell}\right)$ and $\bar{R}_{\mu\nu} = R_{\mu\nu} + \bar{\bar{R}}_{\mu\nu}$.

In general, considering non-metricity properties:

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \bar{R}_{\mu\nu}u^\mu u^\nu - 2\sigma^2 + 2\omega^2 + h^{\mu\nu}D_\nu a_\mu + \frac{1}{\ell^2}a_\mu A^\mu - \frac{1}{\ell^2}(a_\mu u^\mu)^{\cdot} - \frac{2\theta}{3\ell^2}a_\mu u^\mu$$

$$+ \frac{2}{3\ell^4}(a_\mu u^\mu)^2 + \frac{2}{\ell^2}a_\mu \xi^\mu - \dot{\tilde{Q}}_\mu u^\mu + \frac{1}{3}\left(\theta + \frac{1}{\ell^2}a_\nu u^\nu\right)(Q_\mu - \tilde{Q}_\mu)u^\mu - Q_{\mu\nu\lambda}(\sigma^{\mu\nu} + \omega^{\mu\nu})u^\lambda$$

$$-\frac{1}{\ell^2}Q_{\mu\nu\lambda}u^\mu u^\nu(a^\lambda + \xi^\lambda) + Q_{\mu\nu\lambda}u^\mu\sigma^{\nu\lambda} + \frac{1}{\ell^2}Q_{\mu\nu\lambda}(u^\mu\xi^\nu + a^\mu u^\nu)u^\lambda + u^\mu u^\nu D^\lambda Q_{\mu\nu\lambda} + Q_\mu{}^{\lambda\beta}Q_{\beta\lambda\nu}u^\mu u^\nu$$

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where $a_\mu = u^\lambda D_\lambda u_\mu$, $A^\mu = u^\lambda D_\lambda u^\mu$, $\xi_\mu = u^\nu D_\mu u_\nu$, $Q_\mu = Q_{\mu\nu}{}^\nu$, and $\tilde{Q}^\mu = Q^\nu{}_{\nu\mu}$.

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- The generalized Raychaudhuri equation contains additional terms arising from non-metricity of the connection;
- Several terms depend explicitly on the ordering of indices; raising and lowering indices does not commute due to non-metricity;

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- The generalized Raychaudhuri equation contains additional terms arising from non-metricity of the connection;
- Several terms depend explicitly on the ordering of indices; raising and lowering indices does not commute due to non-metricity;
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In general, considering non-metricity properties:

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- Several terms depend explicitly on the ordering of indices; raising and lowering indices does not commute due to non-metricity;
- Covariant derivatives acting on tensors must be treated carefully, as different contractions lead to inequivalent contributions;
- In NMCWCG, the non-conservation of energy–momentum tensor induces extra force terms, sourced by non-metricity and non-minimal coupling.

Conclusion

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- The generalized Raychaudhuri equation **reformulates energy conditions** beyond metric compatibility;
- The resulting geometrical constraints allow **constraints on theory's parameters** and assess its physical viability.

Thank You for Your Attention!

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