Solving the Teukolsky Equation with Spectral Methods

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1 Teukolsky Equation

The **Teukolsky equation (TE)** simulates a **Kerr Black Hole (BH)** together with other much smaller bodies in its orbit. This is the so-called **Extreme-Mass Ratio Inspiral (EMRI)** regime. Although this equation is **linearized**, the properties of the Black Hole can be extracted from its solutions.

Instead of metric components, the TE evolves **Weyl scalars**. Thus, **Gravitational perturbations** are fully determined by Ψ_0(ingoing) and Ψ_4(outgoing).

With the emergence of the next generation of **Gravitational Wave detectors** (e.g. **LISA**), there is a need for **accurate** and **efficient solvers** of the EMRI scenario. This is the main motivation behind my work.

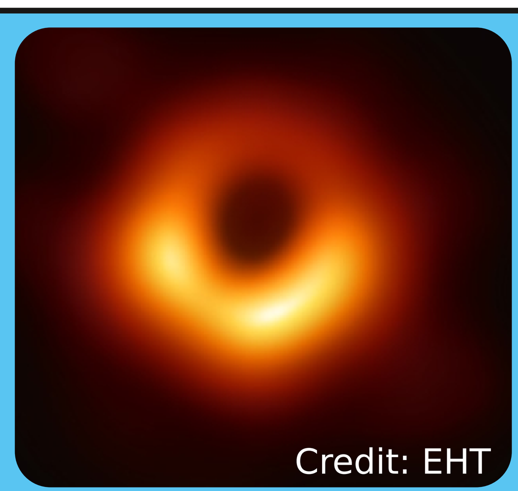


Fig. 1: First image of a BH

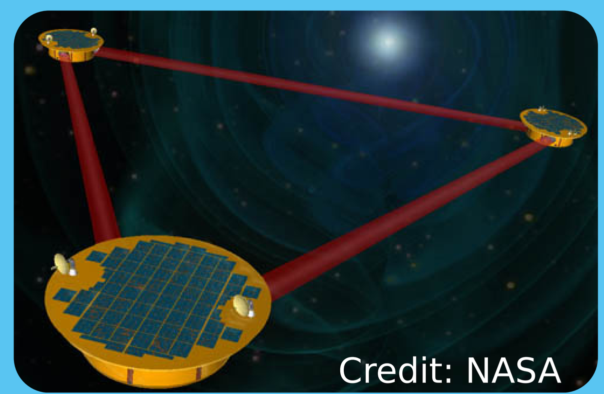


Fig. 2: Illustration of the LISA GW detector

Hyperboloidal Coordinates

How do we numerically simulate **infinite spacetime**? Our code uses **hyperboloidal slices**. There are several advantages to this formalism:

- Boundary conditions are automatically satisfied
- The waveform can be easily extracted at future null infinity, $\mathscr{I}^+$

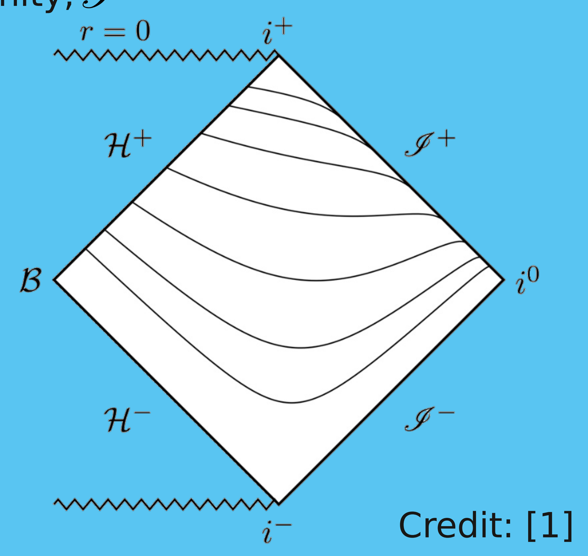


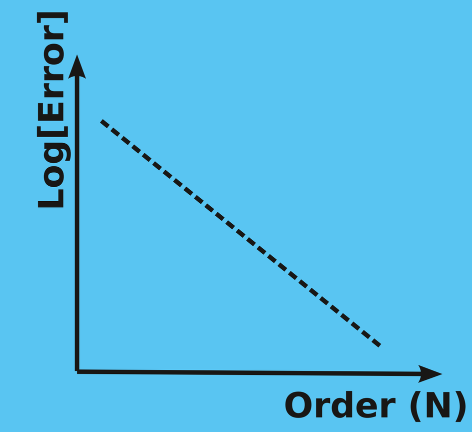
Fig. 3: Penrose diagram of a Schwarzschild BH with hyperboloidal foliation

2 Why Spectral Methods?

Under the right conditions, these numerical schemes show **exponential convergence**: the **numerical error** of solving the equation **decreases exponentially** with the **order** of the method, **N**.

With **spectral methods** one chooses a **set of basis functions** (e.g. Chebyshev Polynomials) and represent the solution in terms of it,

$$\Psi = \sum_{i=0}^{N} a_i T_i$$



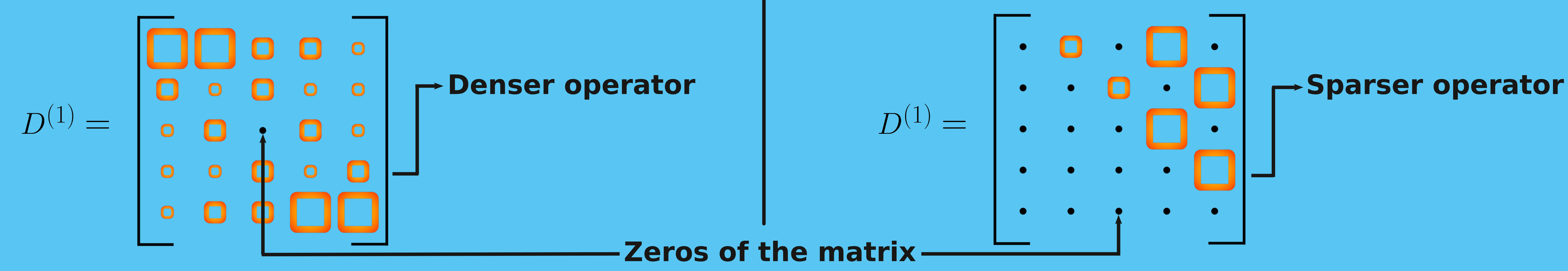
Pseudo-Spectral (PS) → "Fully" Spectral (FS)

In this scheme, we evolve the value of **the field** at the **grid points**. This approach is quite **general** and more efficient that polynomial convergence methods.

Grid points: $x_i = \frac{x_+}{2}\left[1+\sin\left(\frac{2i-N}{2N}\pi\right)\right]$

Now there is **no collocation**, but there are "D" and "X" operators. Still, this scheme shows **less noise** at later times which will be apparent with the results of the next section.

Recurrence formula: $T_{n+1} = 2xT_n - T_{n-1}$



Although we **started** with a **PS scheme** to solve the **homogeneous TE** (showed in **[1]**), afterwards we developed this **FS implementation** due to the aforementioned advantages and we obtained a **more stable** implementation!

Note that, so far, we have **no matter sources** and thus are **not** in the **EMRI regime** and this is the next objective in our work. To do so, we will **adapt** a scheme present in **[2]** that uses **discontinuous PS methods** to evolve the **full TE**.

3 Results

Quasi-Normal Modes are present in the **early stages** of the **ringdown phase** of the merger and they describe the dissipation of the field's modes **after coalescence**. Afterwards, the **late-time decay** of each mode follows a **power law** (check figure 4). These decay rates are shown in figure 5 in the form of **Local Power Indices** which are **known theoretically** through **Price's Law** and they were used as **validation** for both solvers. Note that, in general, the **FS scheme** appears to be **more stable**: there is **less noise / less deviation from the theoretical value** for the same time coordinate. **All** of the **next results** are for a **Kerr BH** with an adimensional **spin** parameter of **1/4**.

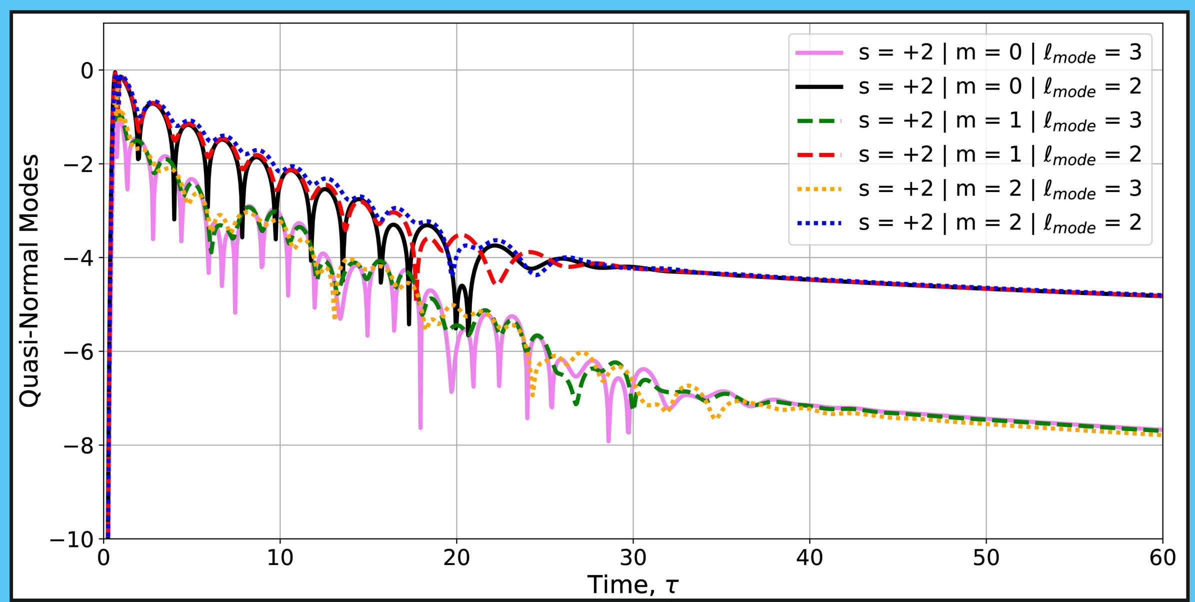


Fig. 4: Quasi-Normal Modes at $\mathscr{I}^+$ for different spin-weights and modes obtained with the FS code

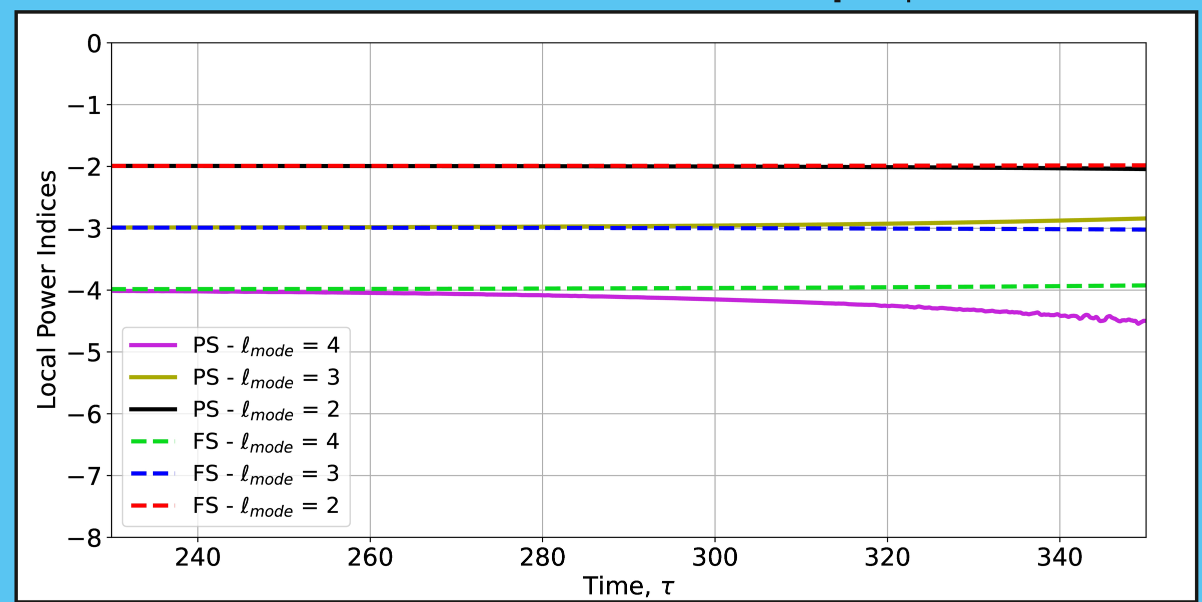


Fig. 5: Comparison of the LPIs at $\mathscr{I}^+$ obtained with the PS and the FS codes

We encountered several **interesting phenomena** of the **TE** with these solvers. Most notably, the **sharp feature** that appears at **future null infinity** for later times (check figure 6) and the case where the decay rate of the **lowest ℓ-mode** starts out at a certain value and slowly **transitions** into another one depending on the amplitudes of the initial signals in 2 **ℓ-modes** (check figure 7).

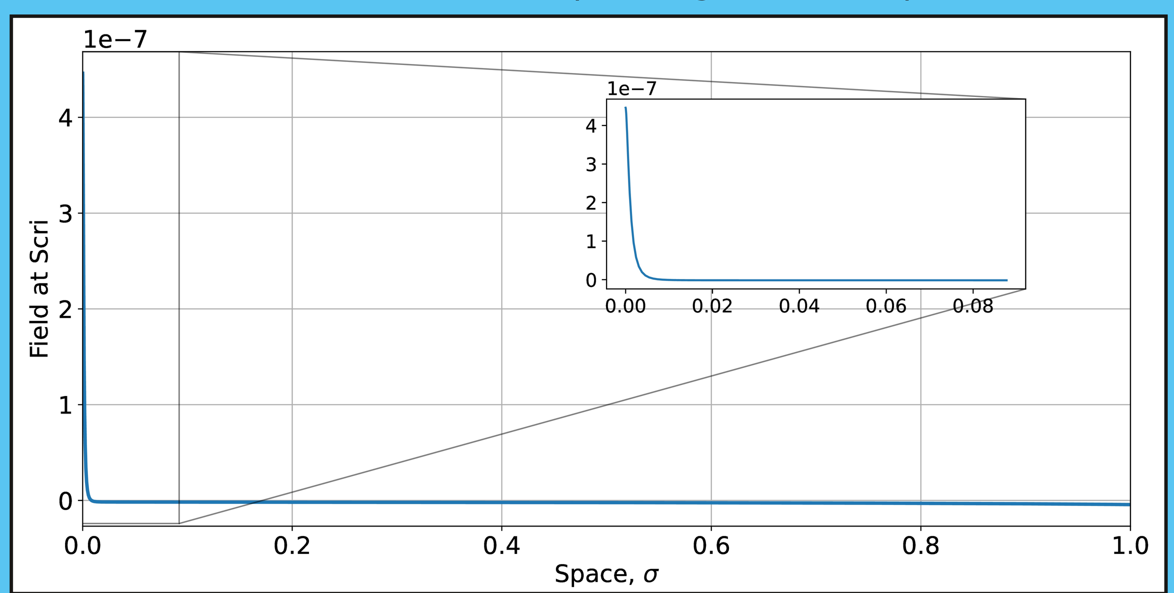


Fig. 6: Lowest ℓ-mode of the field in space for a later time ($\tau = 350$) obtained with the FS code

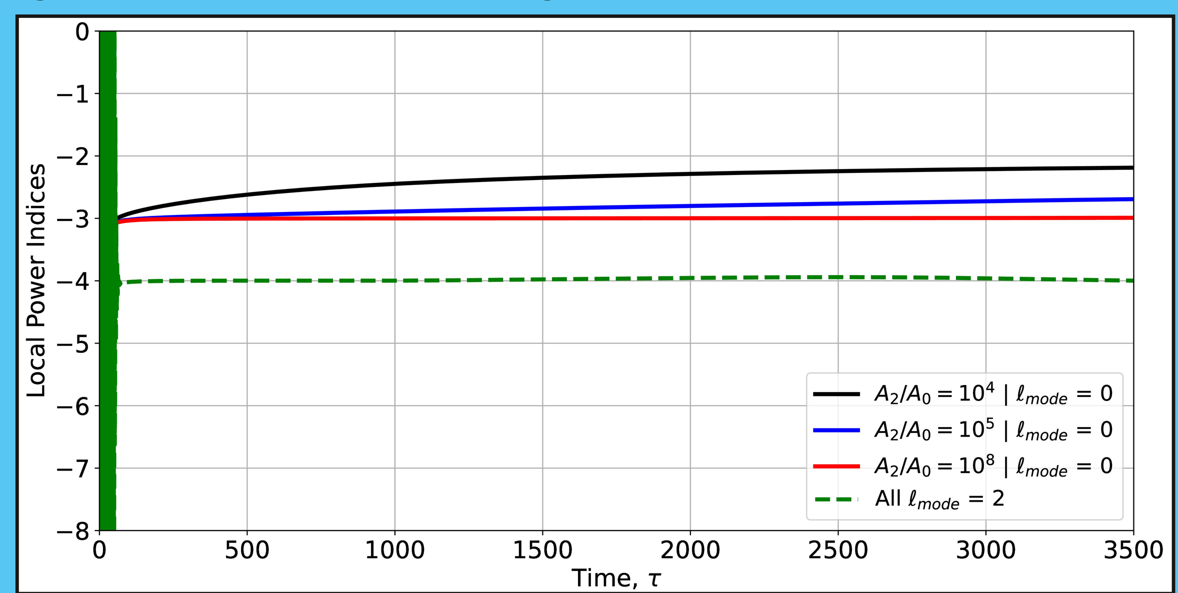


Fig. 7: LPIs at $\mathscr{I}^+$ when 2 ℓ-modes are perturbed with different amplitudes. Used FS code with s = 0 and m = 0

4 Conclusions

- With our work we took an **existing scheme** to solve the TE and we changed the spectral implementation
- This novel implementation was **verified** mainly by the use of **Price's Law**
- **Both** the **algorithms** are **fairly fast** to run on a regular computer
- The **new implementation** resulted in **more accurate / stable** long-time evolutions, which was our goal

Future Work

Throughout this work we have compiled several different directions for **future work:**

- The most obvious one would be to **implement** a spectral scheme to solve the **TE with source terms** (as in **[2]**) in order to evolve **EMRI** scenarios
- It could also be interesting to devise a numerical scheme that deals with the **sharp feature** that appears at **future null infinity for later times** in the **homogeneous case**
- Finally, we would like to implement **several domains** in our **numerical scheme** in order to possibly obtain a **more accurate** and **efficient solver**

References:

[1] C. Markakis, S. Bray and A. Zenginoğlu, Symmetric integration of the 1+1 Teukolsky equation on hyperboloidal foliations of Kerr spacetimes, arXiv:2303.08153 (2023)

[2] L. J. G. Da Silva, DiscoTEX 1.0: Discontinuous collocation and implicit-turned-explicit (IMTEX) integration symplectic, symmetric numerical algorithms with higher order jumps for differential equations I: Numerical black hole perturbation theory applications, arXiv:2401.08758 (2024)